Journal of Applied Mathematics and Computing with applications
ISSN: ..., URL: http://www.joamac.com
Volume 2 Issue 1(2021), Pages 58-85.

SOME NEW REFINEMENTS POST QUANTUM TRAPEZIUM LIKE INEQUALITIES PERTAINING TWICE (p,q)–DIFFERENTIABLE CONVEX FUNCTIONS

ROZANA LIKO, ARTION KASHURI

Abstract. In this paper, we establish a new post quantum integral identity pertaining twice (p,q)–differentiable functions. Applying this identity as an auxiliary result, Hölder–İşcan inequality and Improved–Power mean inequality, we obtain some new refinements post quantum inequalities via twice (p,q)–differentiable convex functions.

1. Introduction and preliminaries

Recall that the secant line joining the images of any two points in the domain of a convex function dominates the entire graph between the points. A function $\xi : \Xi \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex on Ξ, if

$$\xi(c\varepsilon + (1-\varepsilon)d) \leq \varepsilon\xi(c) + (1-\varepsilon)\xi(d), \tag{1.1}$$

holds for all c, $d \in \Xi$, $c < d$ and $\varepsilon \in [0,1]$. We say that ξ is concave, if the inequality (1.1) holds in the reverse direction. Following midpoint inequality is the particular case of a convex function ξ.

$$\xi\left(\frac{c+d}{2}\right) \leq \frac{\xi(c)+\xi(d)}{2}. \tag{1.2}$$

One of the marvelous result in the theory of inequalities, is the Hermite–Hadamard inequality or known as trapezium inequality (HHI) which gives a refinement of (1.2). The HHI is viewed as a criterion for convex functions. For c, $d \in \Xi$, $c < d$, the following double inequality:

$$\xi\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \xi(\varsigma)d\varsigma \leq \frac{\xi(c)+\xi(d)}{2}, \tag{1.3}$$

holds for any convex function $\xi : \Xi \to \mathbb{R}$. It not only provide an estimate of the average integral of a given convex function ξ but also gives a refinement to the famous Jensen inequality. The geometrical significance attracts the researcher to obatin the average integrals for generalized classes of convex functions. So far,

2000 *Mathematics Subject Classification.* 26A51, 05A30, 26A33, 26D07, 26D10, 26D15.
Key words and phrases. Hermite–Hadamard inequality; convexity; (p,q)–calculus; Hölder–İşcan inequality; Improved–Power mean inequality.

Communicated by Naim L. Braha.

many generalizations and applications has become the part of literature, see [1, 2, 3, 4, 5, 6, 7, 8, 9] and the references therein.

The study of quantum calculus (more briefly q–calculus) is the calculus with no limits. Famous mathematician Euler was responsible for introducing q as a parameter in the Newton's work of infinite series. Jackson worked on the classical definition of the derivative of a function and proved many results in the domain of q–calculus. Jackson got the credit for a systematic development of q–calculus. We refer to [10, 11]. The mentioned branch is considered as an incorporative subject between physics and mathematics, as it has numerous applications in quantum theory, mechanics, theory of relativity, orthogonal polynomials, number theory, combinatorics, hypergeometric functions, etc. For some current advancement in this field, we refer to [12, 13].

For a real function ξ, the q–derivative is characterized by

$$D_q\xi(\varsigma) = \frac{\xi(q\varsigma) - \xi(\varsigma)}{(q-1)\varsigma}, \tag{1.4}$$

where $q \in (0,1)$. Numerically, it gives the slope of the line joining $(q\varsigma, \xi(q\varsigma))$ and $(\varsigma, \xi(\varsigma))$. The Jackson integral of a real function ξ is defined by the following series expansion

$$\int_0^{\mu} \xi(\varsigma) d_q\varsigma = (1-q)\mu \sum_{r=0}^{\infty} q^r \xi(q^r\mu). \tag{1.5}$$

The authors in [14, 15], studied the notion of q-derivative and q-integral over the finite real interval $[c,d]$ and defined the q_c-derivative and q_c-integral. For a function $\xi : [c,d] \to \mathbb{R}$ and $q \in (0,1)$, the q_c-derivative and q_c-integral of ξ at $\varsigma \in [c,d]$ are defined and denoted by

$${}_cD_q\xi(\varsigma) = \frac{\xi(\varsigma) - \xi(q\varsigma + (1-q)c)}{(1-q)(\varsigma - c)}, \quad \varsigma \neq c \tag{1.6}$$

and

$$\int_c^{\mu} \xi(\varsigma) {}_cd_q\varsigma = (1-q)(\mu - c) \sum_{r=0}^{\infty} q^r \xi(q^r\mu + (1-q^r)c), \quad \mu \in [c,d], \tag{1.7}$$

respectively.

The authors in [17, 18, 19, 16], have contributed to the ongoing research in q–calculus and have established some interesting results.

Post-quantum or (p,q)–calculus [20, 21, 22, 23, 24] is the generalization of q–calculus. It is natural to explore the behaviour of real functions in the premises of (p,q)–calculus.

In the recently past, the authors in [25], studied the concept of $(p,q)_c$–derivative and $(p,q)_c$-integral over the real interval $[c,d]$.

Definition 1.1. [25, 26] For a function $\xi : [c,d] \to \mathbb{R}$ and $0 < q < p \leq 1$, then $(p,q)_c$-derivative of ξ at $\varsigma \in [c,d]$ is expressed as

$${}_cD_{p,q}\xi(\varsigma) = \frac{\xi(p\varsigma + (1-p)c) - \xi(q\varsigma + (1-q)c)}{(p-q)(\varsigma - c)}, \quad \varsigma \neq c. \tag{1.8}$$

The function ξ is called $(p,q)_c$–differentiable on $[c,d]$, if ${}_cD_{p,q}\xi(\varsigma)$ exists for all $\varsigma \in [c,d]$. It is evident that

$$ {}_cD_{p,q}\xi(c) = \lim_{\varsigma \to c} {}_cD_{p,q}\xi(\varsigma). \tag{1.9}$$

Clearly, if $p=1$ in (1.8), then we have q_c–derivative (1.6). If $c=0$, then we get (p,q)–derivative in the classical sense as follows (see [27]):

$$ {}_0D_{p,q}\xi(\varsigma) = D_{p,q}\xi(\varsigma) = \frac{\xi(p\varsigma)-\xi(q\varsigma)}{(p-q)\varsigma}, \quad \varsigma \neq 0. \tag{1.10}$$

The function ξ is called twice $(p,q)_c$–differentiable on $[c,d]$, if ${}_cD^2_{p,q}\xi(\varsigma)$ exists for all $\varsigma \in [c,d]$ and is defined as

$$ {}_cD^2_{p,q}\xi(\varsigma) = {}_cD_{p,q}\left({}_cD_{p,q}\xi(\varsigma)\right). \tag{1.11}$$

Definition 1.2. [26, 25] Let $\xi : [c,d] \to \mathbb{R}$ be a function and $0<q<p\leq 1$. The definite $(p,q)_c$–integral of the function ξ is characterized by the expression

$$ \int_c^{\mu} \xi(\varsigma)\,{}_cd_{p,q}\varsigma = (p-q)(\mu-c)\sum_{r=0}^{\infty}\frac{q^r}{p^{r+1}}\xi\left(\frac{q^r}{p^{r+1}}\mu + \left(1-\frac{q^r}{p^{r+1}}\right)c\right), \quad \mu \in [c,d]. \tag{1.12}$$

Remark 1.3. (i) In Definition 1.2, if $p=1$, then we have (1.7). If $c=0$, then we get (p,q)–integral in the classical sense (see [27]):

$$ \int_0^{\mu} \xi(\varsigma)\,{}_0d_{p,q}\varsigma = \int_0^{\mu} \xi(\varsigma)\,d_{p,q}\varsigma = (p-q)\mu\sum_{r=0}^{\infty}\frac{q^r}{p^{r+1}}\xi\left(\frac{q^r}{p^{r+1}}\mu\right), \quad \mu \in [c,d]. \tag{1.13}$$

(ii) In Definition 1.2, if $c=0$ and $\mu=1$, then we obtain (see [27]):

$$ \int_0^{1} \xi(\varsigma)\,{}_0d_{p,q}\varsigma = (p-q)\sum_{r=0}^{\infty}\frac{q^r}{p^{r+1}}\xi\left(\frac{q^r}{p^{r+1}}\right). \tag{1.14}$$

Definition 1.4. [27] For any real number, the (p,q)–bracket or twin-basic number is defined as

$$ [n]_{p,q} = \frac{p^n-q^n}{p-q} = p^{n-1}+p^{n-2}q+p^{n-3}q^2+\ldots+pq^{n-2}+q^{n-1}. \tag{1.15}$$

The twin-basic number is a natural generalization of the q–number, that is

$$ \lim_{p\to 1^-}[n]_{p,q} = [n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\ldots+q^{n-2}+q^{n-1}. \tag{1.16}$$

Taking $q \to 1^-$, then

$$ \lim_{q\to 1^-}[n]_q = 1+1+1+\ldots+1 = n. \tag{1.17}$$

We see that, $[n]_{p,q}$ plays the same role in (p,q)–calculus as the integer n in classical calculus and its q–analogue does in q–calculus.

The following (p,q)–integrals for suitable choices of function ζ will be very useful in our main results. For calculating them, we will use Definition 1.2 for special values $c=0$ and $\mu=1$. Interested reader can see also equality (1.14).

Lemma 1.5. (1) *Taking* $\zeta(u) = u$, *then*

$$\int_0^1 u d_{p,q}u = (p-q)\sum_{m=0}^{\infty}\frac{q^{2m}}{p^{2m+2}} = \frac{p-q}{p^2}\sum_{m=0}^{\infty}\left(\frac{q}{p}\right)^{2m} = \frac{p-q}{p^2}\left[\frac{1}{1-\frac{q^2}{p^2}}\right] = \frac{1}{p+q}. \tag{1.18}$$

(2) *Choosing* $\zeta(u) = u^2$, *then*

$$\int_0^1 u^2 d_{p,q}u = (p-q)\sum_{m=0}^{\infty}\frac{q^{3m}}{p^{3m+3}} = \frac{p-q}{p^3}\sum_{m=0}^{\infty}\left(\frac{q}{p}\right)^{3m} = \frac{p-q}{p^3}\left[\frac{1}{1-\frac{q^3}{p^3}}\right] = \frac{1}{p^2+pq+q^2}. \tag{1.19}$$

(3) *Taking* $\zeta(u) = 1-u$, *then*

$$\int_0^1 (1-u)d_{p,q}u = \int_0^1 d_{p,q}u - \int_0^1 u d_{p,q}u = 1 - \frac{1}{p+q} = \frac{p+q-1}{p+q}. \tag{1.20}$$

(4) *Choosing* $\zeta(u) = u(1-u)$, *then*

$$\begin{aligned}\int_0^1 u(1-u)d_{p,q}u &= \int_0^1 u d_{p,q}u - \int_0^1 u^2 d_{p,q}u = \frac{1}{p+q} - \frac{1}{p^2+pq+q^2} \\ &= \frac{p^2+pq+q^2-p-q}{(p+q)(p^2+pq+q^2)}.\end{aligned} \tag{1.21}$$

(5) *Taking* $\zeta(u) = u(1-qu)$, *then*

$$\begin{aligned}\int_0^1 u(1-qu)d_{p,q}u &= \int_0^1 u d_{p,q}u - q\int_0^1 u^2 d_{p,q}u = \frac{1}{p+q} - \frac{q}{p^2+pq+q^2} \\ &= \frac{p^2}{(p+q)(p^2+pq+q^2)}.\end{aligned} \tag{1.22}$$

(6) *Choosing* $\zeta(u) = (1-u)(1-qu)$, *then*

$$\begin{aligned}\int_0^1 (1-u)(1-qu)d_{p,q}u &= \int_0^1 (1-u)d_{p,q}u - q\int_0^1 u(1-u)d_{p,q}u \\ &= \frac{p^2(p-1)+pq(p+q)}{(p+q)(p^2+pq+q^2)}.\end{aligned} \tag{1.23}$$

(7) *Taking* $\zeta(u) = u^2(1-qu)$, *then*

$$\begin{aligned}\int_0^1 u^2(1-qu)d_{p,q}u &= \int_0^1 u^2 d_{p,q}u - q\int_0^1 u^3 d_{p,q}u \\ &= \frac{1}{p^2+pq+q^2} - \frac{q}{p^3+p^2q+pq^2+q^3} \\ &= \frac{p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}.\end{aligned} \tag{1.24}$$

(8) *Choosing* $\zeta(u) = u(1-u)(1-qu)$, *then*

$$\begin{aligned}\int_0^1 u(1-u)(1-qu)d_{p,q}u &= \int_0^1 ud_{p,q}u - (1+q)\int_0^1 u^2 d_{p,q}u + q\int_0^1 u^3 d_{p,q}u \\ &= \frac{1}{p+q} - \frac{1+q}{p^2+pq+q^2} + \frac{q}{p^3+p^2q+pq^2+q^3} \\ &= \frac{p^2-p-q}{(p+q)(p^2+pq+q^2)} + \frac{q}{p^3+p^2q+pq^2+q^3} \\ &= \frac{p^4+p^2q^2-p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}.\end{aligned} \tag{1.25}$$

(9) *Taking* $\zeta(u) = u^k(1-u)$, *then*

$$\begin{aligned}\int_0^1 u^k(1-u)d_{p,q}u &= \int_0^1 u^k d_{p,q}u - \int_0^1 u^{k+1}d_{p,q}u \\ &= \frac{p-q}{p^{k+1}-q^{k+1}} - \frac{p-q}{p^{k+2}-q^{k+2}} \\ &= \frac{1}{[k+1]_{p,q}} - \frac{1}{[k+2]_{p,q}}.\end{aligned} \tag{1.26}$$

(10) *Choosing* $\zeta(u) = u^{k+1}(1-qu)$, *then*

$$\begin{aligned}\int_0^1 u^{k+1}(1-qu)d_{p,q}u &= \int_0^1 u^{k+1}d_{p,q}u - q\int_0^1 u^{k+2}d_{p,q}u \\ &= (p-q)\left[\frac{1}{p^{k+2}-q^{k+2}} - \frac{q}{p^{k+3}-q^{k+3}}\right] \\ &= (p-q)\left[\frac{p^{k+2}(p-q)}{(p^{k+2}-q^{k+2})(p^{k+3}-q^{k+3})}\right] \\ &= \frac{p^{k+2}}{[k+2]_{p,q}[k+3]_{p,q}}.\end{aligned} \tag{1.27}$$

(11) *Taking* $\zeta(u) = u^{k+1}(1-u)(1-qu)$, *then*

$$\begin{aligned}&\int_0^1 u^{k+1}(1-u)(1-qu)d_{p,q}u \\ &= \int_0^1 u^{k+1}d_{p,q}u - (1+q)\int_0^1 u^{k+2}d_{p,q}u + q\int_0^1 u^{k+3}d_{p,q}u \\ &= (p-q)\left[\frac{1}{p^{k+2}-q^{k+2}} - \frac{1+q}{p^{k+3}-q^{k+3}} + \frac{q}{p^{k+4}-q^{k+4}}\right] \\ &= (p-q)(p-q)\left[\frac{p^{2k+6}-p^{2k+5}+p^{k+2}q^{k+2}(p-q^{k+2})}{(p^{k+2}-q^{k+2})(p^{k+3}-q^{k+3})(p^{k+4}-q^{k+4})}\right] \\ &= \frac{1}{[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q}}\left[\frac{p^{2k+6}-p^{2k+5}+p^{k+2}q^{k+2}(p-q^{k+2})}{(p-q)}\right].\end{aligned} \tag{1.28}$$

The following Example 1.6 it is important and will be used in sequel.

Example 1.6. Let $\xi(\tau)=\tau^{s_1}(1-\tau)^{s_2}(1-q\tau)^{s_3}$ and denote

$$N(s_1,s_2,s_3;p,q):=\int_0^1 \tau^{s_1}(1-\tau)^{s_2}(1-q\tau)^{s_3}\ d_{p,q}\tau. \tag{1.29}$$

Then

$$N(s_1,s_2,s_3;p,q)=\frac{p-q}{p^{s_1+1}}\sum_{m=0}^{\infty}\left(\frac{q}{p}\right)^{(1+s_1)m}\left(1-\frac{q^m}{p^{m+1}}\right)^{s_2}\left(1-\frac{q^{m+1}}{p^{m+1}}\right)^{s_3}. \tag{1.30}$$

The notation $N(s_1,s_2,s_3;p,q)$ will be used frequently for the different values of s_1,s_2,s_3 in the upcoming results.

The well–known integral inequalities in (p,q)–calculus are given below (see [26]):

Theorem 1.7. ***(Hölder–İşcan inequality)*** *Let $p_1>1$ and $\frac{1}{p_1}+\frac{1}{q_1}=1$ where $0<q<p\leq 1$. If ϕ and ψ are real valued functions on $[c,d]$ and if $|\phi|^{p_1}$ and $|\psi|^{q_1}$ are integrable on $[c,d]$, then*

$$\begin{aligned}\int_c^d |\phi(x)\psi(x)|\ d_{p,q}x &\leq \frac{1}{d-c}\left\{\left(\int_c^d (d-x)\,|\phi(x)|^{p_1}\ d_{p,q}x\right)^{\frac{1}{p_1}}\left(\int_c^d (d-x)\,|\psi(x)|^{q_1}\ d_{p,q}x\right)^{\frac{1}{q_1}}\right.\\ &\quad\left.+\left(\int_c^d (x-c)\,|\phi(x)|^{p_1}\ d_{p,q}x\right)^{\frac{1}{p_1}}\left(\int_c^d (x-c)\,|\psi(x)|^{q_1}\ d_{p,q}x\right)^{\frac{1}{q_1}}\right\}\\ &\leq \left(\int_c^d |\phi(x)|^{p_1}\ d_{p,q}x\right)^{\frac{1}{p_1}}\left(\int_c^d |\psi(x)|^{q_1}\ d_{p,q}x\right)^{\frac{1}{q_1}},\end{aligned} \tag{1.31}$$

where equality holds if and only if $A\,|\phi(x)|^{p_1}=B\,|\psi(x)|^{q_1}$ almost everywhere, where A and B are constants.

Theorem 1.8. ***(Improved–Power mean inequality)*** *Let $q_1\geq 1$ where $0<q<p\leq 1$. If ϕ and ψ are real valued functions on $[c,d]$ and if $|\phi|$ and $|\phi|\,|\psi|^{q_1}$ are integrable on $[c,d]$, then*

$$\begin{aligned}&\int_c^d |\phi(x)\psi(x)|\ d_{p,q}x\\ &\leq \frac{1}{d-c}\left\{\left(\int_c^d (d-x)\,|\phi(x)|\ d_{p,q}x\right)^{1-\frac{1}{q_1}}\left(\int_c^d (d-x)\,|\phi(x)|\,|\psi(x)|^{q_1}\ d_{p,q}x\right)^{\frac{1}{q_1}}\right.\\ &\quad\left.+\left(\int_c^d (x-c)\,|\phi(x)|\ d_{p,q}x\right)^{1-\frac{1}{q_1}}\left(\int_c^d (x-c)\,|\phi(x)|\,|\psi(x)|^{q_1}\ d_{p,q}x\right)^{\frac{1}{q_1}}\right\}\\ &\leq \left(\int_c^d |\phi(x)|\ d_{p,q}x\right)^{1-\frac{1}{q_1}}\left(\int_c^d |\phi(x)|\,|\psi(x)|^{q_1}\ d_{p,q}x\right)^{\frac{1}{q_1}}.\end{aligned} \tag{1.32}$$

Motivated from above results the aim of this paper is to establish a new post quantum integral identity pertaining twice (p,q)–differentiable functions. Applying this identity as an auxiliary result, Hölder–İşcan inequality and Improved-Power mean inequality, we will obtain some new refinements post quantum inequalities via twice (p,q)–differentiable convex functions.

2. Main results

Lemma 2.1. *Let $\xi : \mathbf{J} = [c,d] \subset \mathbb{R} \to \mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ (interior of $\mathbf{J}$) with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0 < q < p \leq 1$. Then the following integral identity holds:*

$$\frac{q\xi(c) + p\xi(pd + (1-p)c)}{p+q} - \frac{1}{p^2(d-c)} \int_c^{p^2d+(1-p^2)c} \xi(u)\ {}_cd_{p,q}u$$
$$= \frac{pq^2(d-c)^2}{p+q} \int_0^1 \tau(1-q\tau)\ {}_cD^2_{p,q}\xi((1-\tau)c + \tau d)\ d_{p,q}\tau. \tag{2.1}$$

Proof. From Definition 1.1, we have

$$\begin{aligned}
&{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\\
&{}_cD_{p,q}({}_cD_{p,q}\xi((1-\tau)c+\tau d))\\
&= \frac{{}_cD_{p,q}\xi((1-p\tau)c+p\tau d) - {}_cD_{p,q}\xi((1-q\tau)c+q\tau d)}{(p-q)(d-c)\tau}\\
&= \frac{1}{(p-q)(d-c)\tau}\left\{\frac{\xi((1-p^2\tau)c+p^2\tau d) - \xi((1-pq\tau)c+pq\tau d)}{(p-q)(d-c)p\tau}\right.\\
&\left. - \frac{\xi((1-pq\tau)c+pq\tau d) - \xi((1-q^2\tau)c+q^2\tau d)}{(p-q)(d-c)q\tau}\right\}\\
&= \frac{1}{pq(p-q)^2(d-c)^2\tau^2}\left\{q\xi((1-p^2\tau)c+p^2\tau d) - (p+q)\xi((1-pq\tau)c+pq\tau d)\right.\\
&\left. + p\xi((1-q^2\tau)c+q^2\tau d)\right\}.
\end{aligned}$$

From Definition 1.2, we get

$$\int_0^1 p\tau(1-q\tau)_c D^2_{p,q}\xi(\tau)\ d_{p,q}\tau$$

$$= \int_0^1 (p\tau - pq\tau^2)\left[\frac{q\xi((1-p^2\tau)c+p^2\tau d)-(p+q)\xi((1-pq\tau)c+pq\tau d)+p\xi((1-q^2\tau)c+q^2\tau d)}{pq(p-q)^2(d-c)^2\tau^2}\right]$$

$$= p\int_0^1 \left[\frac{q\xi((1-p^2\tau)c+p^2\tau d)-(p+q)\xi((1-pq\tau)c+pq\tau d)+p\xi((1-q^2\tau)c+q^2\tau d)}{pq(p-q)^2(d-c)^2\tau}\right] d_{p,q}\tau$$

$$- pq\int_0^1 \left[\frac{q\xi((1-p^2\tau)c+p^2\tau d)-(p+q)\xi((1-pq\tau)c+pq\tau d)+p\xi((1-q^2\tau)c+q^2\tau d)}{pq(p-q)^2(d-c)^2}\right]$$

$$= p\left[q(p-q)\sum_{j=0}^{\infty}\frac{\xi\left((1-\frac{q^j}{p^{j-1}})c+\frac{q^j}{p^{j-1}}d\right)}{pq(p-q)^2(d-c)^2} - (p+q)(p-q)\sum_{j=0}^{\infty}\frac{\xi\left((1-\frac{q^{j+1}}{p^j})c+\frac{q^{j+1}}{p^j}d\right)}{pq(p-q)^2(d-c)^2}\right.$$

$$\left. + p\ (p-q)\sum_{j=0}^{\infty}\frac{\xi\left((1-\frac{q^{j+2}}{p^{j+1}})c+\frac{q^{j+2}}{p^{j+1}}d\right)}{pq(p-q)^2(d-c)^2}\right]$$

$$- pq\left[qp^2(p-q)(d-c)\sum_{j=0}^{\infty}\frac{\frac{q^j}{p^{j+1}}\xi\left((1-\frac{q^j}{p^{j-1}})c+\frac{q^j}{p^{j-1}}d\right)}{p^3q(p-q)^2(d-c)^3}\right.$$

$$- (p+q)p^2(p-q)(d-c)\sum_{j=0}^{\infty}\frac{\frac{q^{j+1}}{p^{j+2}}\xi\left((1-\frac{q^{j+1}}{p^j})c+\frac{q^{j+1}}{p^j}d\right)}{p^2q^2(p-q)^2(d-c)^3}$$

$$\left. + p\ p^2(p-q)(d-c)\sum_{j=0}^{\infty}\frac{\frac{q^{j+2}}{p^{j+3}}\xi\left((1-\frac{q^{j+2}}{p^{j+1}})c+\frac{q^{j+2}}{p^{j+1}}d\right)}{pq^3(p-q)^2(d-c)^3}\right]$$

$$= p\left[q\left(\frac{\sum_{j=0}^{\infty}\xi\left((1-\frac{q^j}{p^{j-1}})c+\frac{q^j}{p^{j-1}}d\right)-\sum_{j=0}^{\infty}\xi\left((1-\frac{q^{j+1}}{p^j})c+\frac{q^{j+1}}{p^j}d\right)}{pq(p-q)(d-c)^2}\right)\right.$$

$$\left. - p\left(\frac{\sum_{j=0}^{\infty}\xi\left((1-\frac{q^{j+1}}{p^j})c+\frac{q^{j+1}}{p^j}d\right)-\sum_{j=0}^{\infty}\xi\left((1-\frac{q^{j+2}}{p^{j+1}})c+\frac{q^{j+2}}{p^{j+1}}d\right)}{pq(p-q)(d-c)^2}\right)\right]$$

$$- pq\left[\frac{q\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u}{p^3q(p-q)^2(d-c)^3}\right.$$

$$- \frac{(p+q)\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u-(p+q)(p-q)(d-c)p\xi(pd+(1-p)c)}{p^2q^2(p-q)^2(d-c)^3}$$

$$\left. + \frac{p\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u-(p-q)(d-c)p^2\xi(pd+(1-p)c)-pq(p-q)(d-c)q\xi(qd+(1-q)c)}{pq^3(p-q)^2(d-c)^3}\right]$$

$$= p\left[\left(\frac{\xi(pd+(1-p)c)-\xi(c)}{pq(d-c)}\right)-\left(\frac{\xi(qd+(1-q)c)-\xi(c)}{pq(d-c)}\right)\right]$$

Multiplying the last equality by $\frac{q^2(d-c)^2}{p+q}$, we obtain the desired equality (2.1). $\square$

Remark 2.2. Taking $p=1$ in Lemma 2.1, we get ([17], Lemma 4.1).

Theorem 2.3. *Let $\xi : \mathbf{J} = [c,d] \subset \mathbb{R} \to \mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ and $c,d \in \mathbf{J}^\circ$ with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0<q<p\leq 1$. If $\left|{}_cD^2_{p,q}\xi\right|^k$ is convex on $[c,d]$ for $k\geq 1$, then*

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|\leq\frac{pq^2(d-c)^2}{p+q}$$
$$\times\left[\left(\frac{p^2+pq+q^2-p-q}{(p+q)(p^2+pq+q^2)}\right)^{1-\frac{1}{k}}\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(2,1,k;p,q)+\left|{}_cD^2_{p,q}\xi\,(c)\right|^k N(1,2,k;p,q)\right)^{\frac{1}{k}}\right.$$
$$\left.+\left(\frac{1}{p^2+pq+q^2}\right)^{1-\frac{1}{k}}\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(3,0,k;p,q)+\left|{}_cD^2_{p,q}\xi\,(c)\right|^k N(2,1,k;p,q)\right)^{\frac{1}{k}}\right], \tag{2.3}$$

where $N(2,1,k;p,q)$, $N(1,2,k;p,q)$ and $N(3,0,k;p,q)$ are obtained in Example 1.6.

Proof. From Lemma 2.1, applying property of modulus, we have

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\int_0^1\tau(1-q\tau)\ \left|{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\right|^k\ d_{p,q}\tau. \tag{2.4}$$

By convexity of $\left|{}_cD^2_{p,q}\xi\right|^k$ and Improved–Power mean inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|\ d_{p,q}\tau \\
&\leq \left(\int_0^1 (1-\tau)\tau d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 (1-\tau)\tau(1-q\tau)^k\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+\left(\int_0^1 \tau^2 d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 \tau^2(1-q\tau)^k\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&\leq \left(\int_0^1 (1-\tau)\tau d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 \tau(1-\tau)(1-q\tau)^k\left[\tau\left|{}_cD^2_{p,q}\xi(d)\right|^k+(1-\tau)\left|{}_cD^2_{p,q}\xi\,(c)\right|^k\right]d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+\left(\int_0^1 \tau^2 d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 \tau^2(1-q\tau)^k\left[\tau\left|{}_cD^2_{p,q}\xi(d)\right|^k+(1-\tau)\left|{}_cD^2_{p,q}\xi\,(c)\right|^k\right]d_{p,q}\tau\right)^{\frac{1}{k}} \\
&=\left(\int_0^1 (1-\tau)\tau d_{p,q}\tau\right)^{1-\frac{1}{k}} \\
&\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\int_0^1 \tau^2(1-\tau)(1-q\tau)^k d_{p,q}\tau+\left|{}_cD^2_{p,q}\xi\,(c)\right|^k\int_0^1 \tau(1-\tau)^2(1-q\tau)^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+\left(\int_0^1 \tau^2 d_{p,q}\tau\right)^{1-\frac{1}{k}} \\
&\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\int_0^1 \tau^3(1-q\tau)^k d_{p,q}\tau+\left|{}_cD^2_{p,q}\xi\,(c)\right|^k\int_0^1 \tau^2(1-\tau)(1-q\tau)^k d_{p,q}\tau\right)^{\frac{1}{k}}, \qquad (2.5)
\end{aligned}
$$

Utilizing results (1.19) and (1.21) from Lemma 1.5, Example 1.6 and inequalities (2.4), (2.5), we obtain the desired inequality (2.3). □

Theorem 2.4. *Let $\xi:\mathbf{J}=[c,d]\subset\mathbb{R}\to\mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ and $c,d\in\mathbf{J}^\circ$ with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0<q<p\leq 1$. If $\left|{}_cD^2_{p,q}\xi\right|^k$ is convex on $[c,d]$ for $k\geq 1$, then*

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\left[\left(\frac{p^4+p^2q^2-p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)^{1-\frac{1}{k}}\right.$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\frac{1}{[3]_{p,q}[4]_{p,q}[5]_{p,q}}\left[\frac{p^8-p^7+p^3q^3(p-q^3)}{p-q}\right]+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(1,2,1;p,q)\right)^{\frac{1}{k}}$$
$$+\left(\frac{p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)^{1-\frac{1}{k}}$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\left(\frac{p^4}{(p^4+p^3q+p^2q^2+pq^3+q^4)(p^3+p^2q+pq^2+q^3)}\right)\right.$$
$$\left.\left.+\ \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k\frac{1}{[3]_{p,q}[4]_{p,q}[5]_{p,q}}\left[\frac{p^8-p^7+p^3q^3(p-q^3)}{p-q}\right]\right)^{\frac{1}{k}}\right], \tag{2.6}$$

Proof. From Lemma 2.1, applying property of modulus, we have

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\int_0^1\tau(1-q\tau)\ \left|{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\right|^k\ d_{p,q}\tau. \tag{2.7}$$

By convexity of $\left|{}_cD^2_{p,q}\xi\right|^k$ and Improved–Power mean inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|\ d_{p,q}\tau \\
&\leq \left(\int_0^1 (1-\tau)\tau(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 (1-\tau)\tau(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau^2(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 \tau^2(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&\leq \left(\int_0^1 (1-\tau)\tau(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \\
&\times \left(\int_0^1 \tau(1-\tau)(1-q\tau)\left[\tau\left|{}_cD^2_{p,q}\xi(d)\right|^k+(1-\tau)\left|{}_cD^2_{p,q}\xi\,(c)\right|^k\right]d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau^2(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 \tau^2(1-q\tau)\left[\tau\left|{}_cD^2_{p,q}\xi(d)\right|^k+(1-\tau)\left|{}_cD^2_{p,q}\xi\,(c)\right|^k\right]d_{p,q}\tau\right)^{\frac{1}{k}} \\
&= \left(\int_0^1 (1-\tau)\tau(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\int_0^1 \tau^2(1-\tau)(1-q\tau)d_{p,q}\tau+\left|{}_cD^2_{p,q}\xi\,(c)\right|^k\int_0^1 \tau(1-\tau)^2(1-q\tau)d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau^2(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\int_0^1 \tau^3(1-q\tau)+\left|{}_cD^2_{p,q}\xi\,(c)\right|^k\int_0^1 \tau^2(1-\tau)(1-q\tau)d_{p,q}\tau\right)^{\frac{1}{k}}. \qquad (2.8)
\end{aligned}
$$

Utilizing results (1.25) and (1.28) from Lemma 1.5, Example 1.6 and inequalities (2.7), (2.8), we obtain the required inequality (2.6). □

Theorem 2.5. *Let $\xi:\mathbf{J}=[c,d]\subset\mathbb{R}\to\mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ and $c,d\in\mathbf{J}^\circ$ with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0<q<p\leq 1$. If $\left|{}_cD^2_{p,q}\xi\right|^k$ is convex on $[c,d]$ for $k\geq 1$, then*

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\left[\left(\frac{p^2(p-1)+pq(p+q)}{(p+q)(p^2+pq+q^2)}\right)^{1-\frac{1}{k}}\right.$$
$$\times\left(\frac{\left|{}_cD^2_{p,q}\xi(d)\right|^k}{[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q}}\left[\frac{p^{2k+6}-p^{2k+5}+p^{k+2}q^{k+2}(p-q^{k+2})}{(p-q)}\right]+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(k,2,1;p;q)\right)^{\frac{1}{k}}$$
$$+\left(\frac{p^2}{(p+q)(p^2+pq+q^2)}\right)^{1-\frac{1}{k}}$$
$$\left.\times\left(\frac{p^{k+3}\left|{}_cD^2_{p,q}\xi(d)\right|^k}{[k+3]_{p,q}[k+4]_{p,q}}+\frac{\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k}{[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q}}\left[\frac{p^{2k+6}-p^{2k+5}+p^{k+2}q^{k+2}(p-q^{k+2})}{(p-q)}\right]\right)^{\frac{1}{k}}\right].$$
(2.9)

Proof. From Lemma 2.1, applying property of modulus, we have

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\int_0^1\tau(1-q\tau)\ \left|{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\right|^k\ d_{p,q}\tau.$$
(2.10)

By convexity of $\left|{}_cD^2_{p,q}\xi\right|^k$ and Improved–Power mean inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|\ d_{p,q}\tau \\
&\leq \left(\int_0^1 (1-\tau)(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 (1-\tau)\tau^k(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 \tau^{k+1}(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&\leq \left(\int_0^1 (1-\tau)(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^{k+1}(1-\tau)(1-q\tau)d_{p,q}\tau + \int_0^1 \tau^k(1-\tau)^2(1-q\tau)\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau(1-q\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^{k+2}(1-q\tau)d_{p,q}\tau + \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k \int_0^1 \tau^{k+1}(1-q\tau)(1-\tau)d_{p,q}\tau\right)^{\frac{1}{k}} \\
&= \left(\frac{p^2(p-1)+pq(p+q)}{(p+q)(p^2+pq+q^2)}\right)^{1-\frac{1}{k}} \\
&\times \left(\frac{\left|{}_cD^2_{p,q}\xi(d)\right|^k}{[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q}}\left[\frac{p^{2k+6}-p^{2k+5}+p^{k+2}q^{k+2}(p-q^{k+2})}{(p-q)}\right] + \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(k,2,1;p;q)\right)^{\frac{1}{k}} \\
&+ \left(\frac{p^2}{(p+q)(p^2+pq+q^2)}\right)^{1-\frac{1}{k}} \\
&\times \left(\frac{p^{k+3}\left|{}_cD^2_{p,q}\xi(d)\right|^k}{[k+3]_{p,q}[k+4]_{p,q}} + \frac{\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k}{[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q}}\left[\frac{p^{2k+6}-p^{2k+5}+p^{k+2}q^{k+2}(p-q^{k+2})}{(p-q)}\right]\right)^{\frac{1}{k}}, \quad (2.11)
\end{aligned}
$$

Utilizing results (1.22), (1.23), (1.27) and 1.28 from Lemma 1.5, Example 1.6 and inequalities (2.10), (2.11), we obtain the required inequality (2.9). □

Theorem 2.6. *Let $\xi : \mathbf{J} = [c,d] \subset \mathbb{R} \to \mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ and $c,d \in \mathbf{J}^\circ$ with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0 < q < p \leq 1$. If $\left|{}_cD^2_{p,q}\xi\right|^k$ is convex on $[c,d]$ for $k \geq 1$, then*

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\left[\left(\frac{p+q-1}{p+q}\right)^{1-\frac{1}{k}}\right.$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(k+1,1,k;p,q)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(k,2,k;p,q)\right)^{\frac{1}{k}}$$
$$+\left(\frac{1}{p+q}\right)^{1-\frac{1}{k}}$$
$$\left.\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(k+2,0,k;p,q)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(k+1,1,k;p,q)\right)^{\frac{1}{k}}\right]. \quad (2.12)$$

Proof. From Lemma 2.1, applying property of modulus, we have

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\int_0^1\tau(1-q\tau)\ \left|{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\right|^k\ d_{p,q}\tau. \quad (2.13)$$

By convexity of $\left|{}_cD^2_{p,q}\xi\right|^k$ and Improved–Power mean inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|\ d_{p,q}\tau \\
&\leq \left(\int_0^1 (1-\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 (1-\tau)\tau^k(1-q\tau)^k\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+\left(\int_0^1 \tau d_{p,q}\tau\right)^{1-\frac{1}{k}} \left(\int_0^1 \tau^{k+1}(1-q\tau)^k\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&\leq \left(\int_0^1 (1-\tau)d_{p,q}\tau\right)^{1-\frac{1}{k}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^{k+1}(1-\tau)(1-q\tau)^k d_{p,q}\tau + \int_0^1 \tau^k(1-\tau)^2(1-q\tau)^k\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+\left(\int_0^1 \tau d_{p,q}\tau\right)^{1-\frac{1}{k}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^{k+2}(1-q\tau)^k d_{p,q}\tau + \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k \int_0^1 \tau^{k+1}(1-\tau)(1-q\tau)^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&= \left(\frac{p+q-1}{p+q}\right)^{1-\frac{1}{k}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(k+1,1,k;p,q) + \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(k,2,k;p,q)\right)^{\frac{1}{k}} \\
&+\left(\frac{1}{p+q}\right)^{1-\frac{1}{k}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(k+2,0,k;p,q) + \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(k+1,1,k;p,q)\right)^{\frac{1}{k}}.
\end{aligned}
$$

Utilizing results (1.18) and (1.20) from Lemma 1.5, Example 1.6 and inequalities (2.13), (2.14), we obtain the required inequality (2.12). □

Theorem 2.7. *Let $\xi : \mathbf{J} = [c,d] \subset \mathbb{R} \to \mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ and $c,d \in \mathbf{J}^\circ$ with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0 < q < p \leq 1$. If $\left|{}_cD^2_{p,q}\xi\right|^k$ is convex on $[c,d]$ for $k > 1$ with $\frac{1}{s}+\frac{1}{k}=1$, then*

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\left[(N(1,1,s;p,q))^{\frac{1}{s}}\right.$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\left(\frac{pq(p+q-1)-p^2(1-p)-q^2(1-q)}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(1,2,0;p,q)\right)^{\frac{1}{k}}$$
$$+(N(2,0,s;p,q))^{\frac{1}{s}}$$
$$\left.\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\frac{1}{p^3+p^2q+pq^2+q^3}+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k\left(\frac{pq(p+q-1)-p^2(1-p)-q^2(1-q)}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)\right)^{\frac{1}{k}}\right].$$
(2.14)

Proof. From Lemma 2.1, applying property of modulus, we have

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\int_0^1\tau(1-q\tau)\ \left|{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\right|^k\ d_{p,q}\tau.$$
(2.15)

By convexity of $\left|{}_cD^2_{p,q}\xi\right|^k$ and Hölder–İşcan inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau^{\frac{1}{s}+\frac{1}{k}}(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|\ d_{p,q}\tau \\
&\leq \left(\int_0^1 (1-\tau)\tau(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \left(\int_0^1 (1-\tau)\tau\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau^2(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \left(\int_0^1 \tau^2\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&\leq \left(\int_0^1 \tau(1-\tau)(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^2(1-\tau)d_{p,q}\tau + \int_0^1 \tau(1-\tau)^2\left|{}_cD^2_{p,q}\xi\,(c)\right|^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau^2(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^3 d_{p,q}\tau + \left|{}_cD^2_{p,q}\xi\,(c)\right|^k \int_0^1 \tau^2(1-\tau)d_{p,q}\tau\right)^{\frac{1}{k}} \\
&= (N(1,1,s;p,q))^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \left(\frac{pq(p+q-1)-p^2(1-p)-q^2(1-q)}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right) + \left|{}_cD^2_{p,q}\xi\,(c)\right|^k N(1,2,0;p,q)\right)^{\frac{1}{k}} \\
&+ (N(2,0,s;p,q))^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \frac{1}{p^3+p^2q+pq^2+q^3} + \left|{}_cD^2_{p,q}\xi\,(c)\right|^k \left(\frac{pq(p+q-1)-p^2(1-p)-q^2(1-q)}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)\right)^{\frac{1}{k}} \qquad (2.16)
\end{aligned}
$$

Utilizing result (1.26) from Lemma 1.5, Example 1.6 and inequalities (2.15), (2.16), we obtain the required inequality (2.14). □

Theorem 2.8. *Let $\xi : \mathbf{J} = [c,d] \subset \mathbb{R} \to \mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ and $c,d \in \mathbf{J}^\circ$ with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0 < q < p \leq 1$. If $\left|{}_cD^2_{p,q}\xi\right|^k$ is convex on $[c,d]$ for $k > 1$ with $\frac{1}{s}+\frac{1}{k}=1$, then*

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\Bigg[(N(s,1,s;p,q))^{\frac{1}{s}}$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\left(\frac{p^2+pq+q^2-p-q}{(p+q)(p^2+pq+q^2)}\right)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k\left(\frac{p(1-p)^2+q(1-q)^2+2pq(p+q-1)}{(p+q)(p^2+pq+q^2)}\right)\right)^{\frac{1}{k}}$$
$$+(N(s+1,0,s;p,q))^{\frac{1}{s}}$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\left(\frac{1}{p^2+pq+q^2}\right)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k\left(\frac{p^2+pq+q^2-p-q}{(p+q)(p^2+pq+q^2)}\right)\right)^{\frac{1}{k}}\Bigg].\tag{2.17}$$

Proof. From Lemma 2.1, applying property of modulus, we have

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\int_0^1\tau(1-q\tau)\ \left|{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\right|^k\ d_{p,q}\tau.\tag{2.18}$$

By convexity of $\left|{}_cD^2_{p,q}\xi\right|^k$ and Hölder–İşcan inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau^{\frac{1}{s}+\frac{1}{k}}(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|\ d_{p,q}\tau \\
&\leq \left(\int_0^1 (1-\tau)\tau^s(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \left(\int_0^1 (1-\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau^{s+1}(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \left(\int_0^1 \tau\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&\leq \left(\int_0^1 \tau^s(1-\tau)(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau(1-\tau)d_{p,q}\tau + \int_0^1 (1-\tau)^2\left|{}_cD^2_{p,q}\xi\,(c)\right|^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau^{s+1}(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^2 d_{p,q}\tau + \left|{}_cD^2_{p,q}\xi\,(c)\right|^k \int_0^1 \tau(1-\tau)d_{p,q}\tau\right)^{\frac{1}{k}} \\
&= (N(s,1,s;p,q))^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \left(\frac{p^2+pq+q^2-p-q}{(p+q)(p^2+pq+q^2)}\right) + \left|{}_cD^2_{p,q}\xi\,(c)\right|^k \left(\frac{p(1-p)^2+q(1-q)^2+2pq(p+q-1)}{(p+q)(p^2+pq+q^2)}\right)\right)^{\frac{1}{k}} \\
&+ (N(s+1,0,s;p,q))^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \left(\frac{1}{p^2+pq+q^2}\right) + \left|{}_cD^2_{p,q}\xi\,(c)\right|^k \left(\frac{p^2+pq+q^2-p-q}{(p+q)(p^2+pq+q^2)}\right)\right)^{\frac{1}{k}}.
\end{aligned}
$$

Utilizing results (1.19) and (1.22) from Lemma 1.5, Example 1.6 and inequalities (2.18), (2.19), we obtain the required inequality (2.17). □

Theorem 2.9. *Let $\xi : \mathbf{J} = [c,d] \subset \mathbb{R} \to \mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ and $c,d \in \mathbf{J}^\circ$ with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0 < q < p \leq 1$. If $\left|{}_cD^2_{p,q}\xi\right|^k$ is convex on $[c,d]$ for $k > 1$ with $\frac{1}{s}+\frac{1}{k}=1$, then*

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\left[\left(\frac{1}{[s+1]_{p,q}}-\frac{1}{[s+2]_{p,q}}\right)^{\frac{1}{s}}\right.$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(1,1,k;p,q)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(0,2,k;p,q)\right)^{\frac{1}{k}}$$
$$+\left(\frac{1}{[s+2]_{p,q}}\right)^{\frac{1}{s}}$$
$$\left.\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(2,0,k;p,q)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(1,1,k;p,q)\right)^{\frac{1}{k}}\right]. \tag{2.19}$$

Proof. From Lemma 2.1, applying property of modulus, we have

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\int_0^1\tau(1-q\tau)\ \left|{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\right|^k\ d_{p,q}\tau. \tag{2.20}$$

By convexity of $\left|{}_cD^2_{p,q}\xi\right|^k$ and Hölder–İşcan inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|\ d_{p,q}\tau \\
&\leq \left(\int_0^1 (1-\tau)\tau^s d_{p,q}\tau\right)^{\frac{1}{s}} \left(\int_0^1 (1-\tau)(1-q\tau)^k\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau^{s+1} d_{p,q}\tau\right)^{\frac{1}{s}} \left(\int_0^1 \tau(1-q\tau)^k\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&\leq \left(\int_0^1 \tau^s(1-\tau) d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau(1-\tau)(1-q\tau)^k d_{p,q}\tau + \int_0^1 (1-\tau)^2(1-q\tau)^k\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau^{s+1} d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^2(1-q\tau)^k d_{p,q}\tau + \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k \int_0^1 \tau(1-\tau)(1-q\tau)^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&= \left(\frac{1}{[s+1]_{p,q}} - \frac{1}{[s+2]_{p,q}}\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(1,1,k;p,q) + \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(0,2,k;p,q)\right)^{\frac{1}{k}} \\
&+ \left(\frac{1}{[s+2]_{p,q}}\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k N(2,0,k;p,q) + \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(1,1,k;p,q)\right)^{\frac{1}{k}}, \qquad (2.21)
\end{aligned}
$$

Utilizing result (1.26) from Lemma 1.5, Example 1.6 and inequalities (2.20), (2.21), we obtain the required inequality (2.19). □

Theorem 2.10. *Let $\xi : \mathbf{J} = [c,d] \subset \mathbb{R} \to \mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ and $c,d \in \mathbf{J}^\circ$ with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0 < q < p \leq 1$. If $\left|{}_cD^2_{p,q}\xi\right|^k$ is convex on $[c,d]$ for $k > 1$ with $\frac{1}{s} + \frac{1}{k} = 1$, then*

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\Big[(N(0,1,s;p,q))^{\frac{1}{s}}$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\left(\frac{1}{[k+2]_{p,q}}-\frac{1}{[k+3]_{p,q}}\right)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(k,2,0;p,q)\right)^{\frac{1}{k}}$$
$$+(N(1,0,s;p,q))^{\frac{1}{s}}$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\frac{1}{[k+3]_{p,q}}+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k\left(\frac{1}{[k+2]_{p,q}}-\frac{1}{[k+3]_{p,q}}\right)\right)^{\frac{1}{k}}\Big]. \quad (2.22)$$

Proof. From Lemma 2.1, applying property of modulus, we have

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\int_0^1\tau(1-q\tau)\ \left|{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\right|^k\ d_{p,q}\tau. \quad (2.23)$$

By convexity of $\left|{}_cD^2_{p,q}\xi\right|^k$ and Hölder–İşcan inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|\ d_{p,q}\tau \\
&\leq \left(\int_0^1 (1-\tau)(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \left(\int_0^1 (1-\tau)\tau^k\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \left(\int_0^1 \tau^{k+1}\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&\leq \left(\int_0^1 (1-\tau)(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^{k+1}(1-\tau)d_{p,q}\tau + \int_0^1 \tau^k(1-\tau)^2\left|{}_cD^2_{p,q}\xi\,(c)\right|^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+ \left(\int_0^1 \tau(1-q\tau)^s d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \int_0^1 \tau^{k+2} d_{p,q}\tau + \left|{}_cD^2_{p,q}\xi\,(c)\right|^k \int_0^1 \tau^{k+1}(1-\tau)d_{p,q}\tau\right)^{\frac{1}{k}} \\
&= (N(0,1,s;p,q))^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \left(\frac{1}{[k+2]_{p,q}} - \frac{1}{[k+3]_{p,q}}\right) + \left|{}_cD^2_{p,q}\xi\,(c)\right|^k N(k,2,0;p,q)\right)^{\frac{1}{k}} \\
&+ (N(1,0,s;p,q))^{\frac{1}{s}} \\
&\times \left(\left|{}_cD^2_{p,q}\xi(d)\right|^k \frac{1}{[k+3]_{p,q}} + \left|{}_cD^2_{p,q}\xi\,(c)\right|^k \left(\frac{1}{[k+2]_{p,q}} - \frac{1}{[k+3]_{p,q}}\right)\right)^{\frac{1}{k}}. \qquad (2.24)
\end{aligned}
$$

Utilizing result (1.26) from Lemma 1.5, Example 1.6 and inequalities (2.23), (2.24), we obtain the required inequality (2.22). □

Theorem 2.11. *Let $\xi : \mathbf{J} = [c,d] \subset \mathbb{R} \to \mathbb{R}$ be a twice (p,q)–differentiable function on $\mathbf{J}^\circ$ and $c,d \in \mathbf{J}^\circ$ with ${}_cD^2_{p,q}\xi$ be continuous and integrable on $\mathbf{J}$ where $0 < q < p \leq 1$. If $\left|{}_cD^2_{p,q}\xi\right|^k$ is convex on $[c,d]$ for $k > 1$ with $\frac{1}{s} + \frac{1}{k} = 1$, then*

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\left[(N(s,1,1;p,q))^{\frac{1}{s}}\right.$$
$$\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\left(\frac{p^4+p^2q^2-p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(0,2,1;p,q)\right)^{\frac{1}{k}}$$
$$+\left(\frac{p^{s+2}}{[s+2]_{p,q}[s+3]_{p,q}}\right)^{\frac{1}{s}}\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\left(\frac{p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)\right.$$
$$\left.\left.+\ \left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k\left(\frac{p^4+p^2q^2-p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)\right)^{\frac{1}{k}}\right]. \tag{2.25}$$

Proof. From Lemma 2.1, applying property of modulus, we have

$$\left|\frac{q\xi(c)+p\xi(pd+(1-p)c)}{p+q}-\frac{1}{p^2(d-c)}\int_c^{p^2d+(1-p^2)c}\xi(u)\ {}_cd_{p,q}u\right|$$
$$\leq\frac{pq^2(d-c)^2}{p+q}\int_0^1\tau(1-q\tau)\ \left|{}_cD^2_{p,q}\xi((1-\tau)c+\tau d)\right|^k\ d_{p,q}\tau. \tag{2.26}$$

By convexity of $\left|{}_cD^2_{p,q}\xi\right|^k$ and Hölder–İşcan inequality, we get

$$
\begin{aligned}
&\int_0^1 \tau(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|\ d_{p,q}\tau \\
&\leq \left(\int_0^1 (1-\tau)\tau^s(1-q\tau)d_{p,q}\tau\right)^{\frac{1}{s}}\left(\int_0^1 (1-\tau)(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+\left(\int_0^1 \tau^{s+1}(1-q\tau)d_{p,q}\tau\right)^{\frac{1}{s}}\left(\int_0^1 \tau(1-q\tau)\left|\ {}_cD^2_{p,q}\xi(\tau d+(1-\tau)c)\right|^k\ d_{p,q}\tau\right)^{\frac{1}{k}} \\
&\leq \left(\int_0^1 \tau^s(1-\tau)(1-q\tau)d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\int_0^1 \tau(1-\tau)(1-q\tau)d_{p,q}\tau+\int_0^1 (1-\tau)^2(1-q\tau)\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k d_{p,q}\tau\right)^{\frac{1}{k}} \\
&+\left(\int_0^1 \tau^{s+1}(1-q\tau)d_{p,q}\tau\right)^{\frac{1}{s}} \\
&\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\int_0^1 \tau^2(1-q\tau)d_{p,q}\tau+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k\int_0^1 \tau(1-\tau)(1-q\tau)d_{p,q}\tau\right)^{\frac{1}{k}} \\
&=(N(s,1,1;p,q))^{\frac{1}{s}} \\
&\times\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\left(\frac{p^4+p^2q^2-p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k N(0,2,1;p,q)\right)^{\frac{1}{k}} \\
&+\left(\frac{p^{s+2}}{[s+2]_{p,q}[s+3]_{p,q}}\right)^{\frac{1}{s}}\left(\left|{}_cD^2_{p,q}\xi(d)\right|^k\left(\frac{p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)\right. \\
&\left.+\left|{}_cD^2_{p,q}\xi\left(c\right)\right|^k\left(\frac{p^4+p^2q^2-p^3}{(p^2+pq+q^2)(p^3+p^2q+pq^2+q^3)}\right)\right)^{\frac{1}{k}}. \qquad (2.27)
\end{aligned}
$$

Utilizing result (1.24) from Lemma 1.5, Example 1.6 and inequalities (2.26), (2.27), we obtain the required inequality (2.25). □

Remark 2.12. Taking $p,\ q\to 1^-$, then form our main results we can derive many other new and useful inequalities. It is important to mentioned that, for suitable choices of convex function, like $\xi(x)=x^n$, $x>0$, where $n\in\mathbb{N}$ and $n\geq 2$; $\xi(x)=e^x$, $x\in\mathbb{R}$; $\xi(x)=\frac{1}{x}$, $x>0$ etc., we can deduce from our main results many new interesting inequalities using special means. We omit here their proofs due to length of the paper and the details are left to the interested reader.

3. Conclusion

In this paper, we have derived a new post quantum integral identity pertaining twice (p, q)–differentiable functions. Applying this identity as an auxiliary result, Hölder–İşcan inequality and Improved–Power mean inequality, we obtain some new refinements post quantum inequalities via twice (p, q)–differentiable convex functions. Remark 2.12 shown the efficient of our main results regarding application to special means. The results of this paper are new and significantly contribute to the existing literature on the topic. We hope that current work using our idea and technique will attract the attention of researchers working in mathematical analysis and other related fields in pure and applied sciences.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] Dragomir, S.S.; Agarwal, R.P. Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95.
[2] İşcan, İ. Some new Hermite–Hadamard type inequalities for s–convex functions and their applications. *J. Inequal. Appl.* **2019**, *201*.
[3] Kashuri, A.; Liko, R. Some new Hermite–Hadamard type inequalities and their applications. *Stud. Sci. Math. Hung.* **2019**, *56*, 103–142.
[4] Sarikaya, M.Z., Saglam, A.; Yildirim, H. On some Hadamard type inequalities for h–convex functions. *J. Math. Inequal.* **2008**, *2*, 335–341.
[5] İşcan, İ. New refinements for integral and sum forms of Hölder inequality. *J. Inequal. Appl.* **2019**, *304*.
[6] Srivastava, H.M.; Zhang, Z.-H.; Wu, Y.-D. Some further refinements and extensions of the Hermite–Hadamard and Jensen inequalities in several variables. *Math. Comput. Model.* **2011**, *54*, 2709–2717.
[7] Mohammed, P.O.; Abdeljawad, T., Zeng, S.; Kashuri, A. Fractional Hermite–Hadamard integral inequalities for a new class of convex functions. *Symmetry* **2020**, *12*, 1485.
[8] Mohammed, P.O.; Aydi, H., Kashuri, A.; Hamed, Y.S.; Abualnaja, K.M. Midpoint inequalities in fractional calculus defined using positive weighted symmetry function kernels. *Symmetry* **2021**, *13*, 550.
[9] Mohammed, P.O.; Abdeljawad, T.; Kashuri, A. Fractional Hermite-Hadamard-Fejér inequalities for a convex function with respect to an increasing function involving a positive weighted symmetric function. *Symmetry* **2020**, *12*, 1503.
[10] Jackson, F.H. On q–functions and a certain difference operator. *Trans. Roy. Soc. Edin.* **1908**, *46*, 253–281.
[11] Jackson, F.H., On q–definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
[12] Ernst, T. A Comprehensive Treatment of q–Calculus. Springer, Basel 2012.
[13] Gauchman, H. Integral inequalities in q–calculus. *Comput. Math. Appl.* **2004**, *47*, 281-300.
[14] Tariboon, J., Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Difference Equ.* **2013**, *282*, 1–19.
[15] Tariboon, J.; Ntouyas, S.K. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* **2014**, *2014*, 121.
[16] Bermudo, S.; Kórus, P.; Nápoles, J.E.V. On q–Hermite–Hadamard inequalities for general convex functions. *Acta Math. Hungar.* **2020**, `http://doi.org/10.1007/s10474-020-01025-6`.
[17] Liu, W.; Zhuang, H.. Some quantum estimates of Hermite-Hadamard inequalities for convex functions. *J. Appl. Anal. Comput.* **2017**, *7*, 501–522.
[18] Noor, M.A.; Noor, K.I., Awan, M.U. Some quantum estimates for Hermite-Hadamard inequalities. *Appl. Math. Comput.* **2015**, *251*, 675-679.
[19] Sudsutad, W.; Ntouyas, S.K.; Tariboon, J. Quantum integral inequalities for convex functions. *J. Math. Inequal.* **2015**, *9*, 781–793.

[20] Awan, M.U.; Talib, S.; Noor, M.A.; Chu, Y.-M.; Noor, K.I. On post quantum estimates of upper bounds involving twice (p,q)–differentiable preinvex function. *J. Inequal. Appl.* **2020**, *229*, 14.
[21] Kalsoom, H.; Amer, M.; Junjua, M.-D.; Hussain, S.; Shahzadi, G. Some (p,q)–estimates of Hermite–Hadamard-type inequalities for coordinated convex and quasi-convex functions. *Mathematics* **2019**, *7*, 683.
[22] Prabseang, J.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. Refinements of Hermite-Hadamard inequalities for continuous convex functions via (p,q)–calculus. *Mathematics* **2021**, *9*, 446.
[23] Wannalookkhee, F.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. On Hermite–Hadamard type inequalities for coordinated convex functions via (p,q)–calculus. *Mathematics* **2021**, *9*, 698.
[24] Chu, Y.-M.; Talib, S.; Set, E.; Awan, M.U.; Noor, M.A. (p,q)–Analysis of Montgomery identity and estimates of (p,q)–bounds with applications. *J. Inequal. Appl.* **2021**, *9*, 13.
[25] Kunt, M.; İşcan, İ.; Alp, N.; Sarikaya, M.Z. (p,q)–Hermite–Hadamard inequalities and (p,q)–estimates for midpoint type inequalities via convex and quasi convex functions. *RACSAM* **2018**, *112*, 969–992.
[26] Tunç, M.; Göv, G. Some integral inequalities via (p,q)–calculus on finite intervals. *RGMIA Res. Rep. Coll.* **2016**, *19*, 1–13.
[27] Sadjang, P.N. On the fundamental theorem of (p,q)–calculus and some (p,q)–Taylor formulas. *Result. Math.* **2018**, *73*, 39.

Rozana Liko
Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", 9400, Vlora, Albania
E-mail address: rozanaliko86@gmail.com

Artion Kashuri
Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", 9400, Vlora, Albania
E-mail address: artionkashuri@gmail.com