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BOUNDARY COMPONENTS OF FRICKE GROUP

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Abstract. In this study, we deal with an invariant in the signature of a non-Euclidean crystallographic group, a NEC group, for short. We examine the number of boundary components in the signature of Fricke group.

1. Introduction

The concept of the modular group theory has an important role in many areas of mathematics, such as number theory, graph theory, automorphic function theory and combinatorics. Because of this importance, it impresses many matematicians and they take this topic from different angles, such as searching the graph conditions, calculating the normalizer and determinig the boundary components etc., see the references [4, 6, 7, 12]. Moreover, due to their importance in elliptic curves, integral quadratic forms and elliptic modular functions, congruence subgroups of Γ modular group $\Gamma(N)$, $\Gamma_0(N)$, $\Gamma_1(N)$ etc. groups are mostly studied, see the references [1, 2, 9].

Let $\mathbf{H} := \{z \in \mathbf{C} : \mathrm{Im(z)} > 0\}$ denote the upper half plane. With the Poincare metric $ds = |dz|/y$, it becomes a model of the hyperbolic plane. A non-Euclidean crystallographic(NEC) group is a discrete subgroup Λ of the group G of isometries of $\mathbf{H}$ for which $\mathbf{H}/\Lambda$ is compact. If Λ contains only orientation preserving isometries then it is called a Fuchsian group. An NEC group is determined by its signature

$$(g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

If Λ has this signature then $\mathbf{H}/\Lambda$ is an orbifold whose underlying space is a surface of genus g with k boundary components (holes) and it is orientable if the sign $+$ and non-orientable otherwise. There are r cone points of angle $2\pi/m_1, \ldots, 2\pi/m_r$ in the interior of $\mathbf{H}/\Lambda$ and s_i corner points of angle $\pi/n_{i1}, \ldots, \pi/n_{is_1}$ around the ith hole. If the modular group Γ has this signature, then the area of its fundamental region is

$$\mu(\Lambda) = 2\pi\left[\alpha g + k - 2 + \sum_{i=1}^{r}(1 - 1/m_i) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - 1/n_{ij})\right],$$

where $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise.

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2. Preliminaries

We start by recalling definitions and some preliminary results of these concepts. Let PSL(2,$\mathbf{R}$) denote the group of all linear fractional transformations

$$T : z \to \frac{az+b}{cz+d}\text{ , where } a, b, c \text{ and } d \text{ are real and } ad - bc = 1.$$

In terms of matrix representation, the elements of PSL(2,$\mathbf{R}$) correspond to the matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a, b, c, d \in \mathbf{R} \text{ and } ad - bc = 1.$$

This is the automorphism group of the upper half plane $\mathbf{H}$.

The modular group Γ=PSL(2, $\mathbf{Z}$), is the subgroup of PSL(2,$\mathbf{R}$) such that a, b, c and d are integers. It acts on $\hat{\mathbf{H}} := \mathbf{H} \cup \mathbf{Q} \cup \{\infty\}$. The principal congruence subgroup of level N is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} modN \right\}.$$

and any subgroup of Γ containing $\Gamma(N)$ such as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} modN \right\}, \quad \Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} modN \right\}$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} modN \right\}, \quad \Gamma^1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} modN \right\}$$

is called a congruence subgroup of level N.

The standard classification of the elements of $PSL(2, \mathbf{R})$, motivated by their geometric properties when acting on $\mathbf{H}$, depends on the trace of the corresponding matrices. Let G be a subgroup of $PSL(2, \mathbf{R})$. Any point in the boundary of $\mathbf{H}$ fixed by a parabolic element of G is called a parabolic point or cusp of G. The set of the parabolic points of G is called the cusp set of G. Two points x_1, x_2 in the cusp set of G are called G-inequivalent if there is no $g \in G$ such that $gx_1 = x_2$. The number of G-inequivalent cusps of G, that is the number of orbits in the action G on its cusp set, is called the parabolic class number of G.

In this study, we examine an invariant, the number of boundary components, in the signature of a NEC group. Mainly, we will discuss the following NEC group, which is called Fricke group. The group $\Gamma_F(N) =< \Gamma_0(N), \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix} >$ where $N = n^2$ and n is any positive integer.

There are two different ways to calculate the boundary components: Firstly by using the Hoare-Uzzel theorem and secondly by using the technique in H. Jaffee [6]. H. Jaffee found the boundary components by using the fixed points. In [11], Harding investigated the boundary components of the another NEC group $\hat{\Gamma}(N) =< \Gamma(N), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ by using Hoare-Uzzel theorem. In [8], Akbas obtained some results for the group $\hat{\Gamma}_0(N) =< \Gamma_0(N), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} >$ by using either method.

In the Jaffee's tecnique, let X_N denote Shimura's canonical model (to be defined in 5) of the compact Riemann surface $\Gamma(N) \setminus \mathbf{H}^*$, where $\Gamma(N) \subset SL_2(\mathbf{Z})$ is the principal congruence subgroup of level N. They assume that $N \geq 2$. The symbol X_N means a certain nonsingular projective algebraic curve, defined over $\mathbf{Q}$, plus a certain mapping $\phi_N : \mathbf{H}^* \to X_N(\mathbf{C})$ which defines an isomorphism $\Gamma(N) \setminus \mathbf{H}^* \approx X_N(\mathbf{C})^{an}$. A cusp of X_N is the image under ϕ_N of a point $\mathbf{H}^* - \mathbf{H} = \mathbf{Q} \cup \{\infty\}$. The cusps are finite in number and contained in $X_N(K_N)$, where $K_N = \mathbf{Q}(e^{2\pi i/n})$. In the $\mathbf{R}-$ topology, the space $X_N(\mathbf{R})$ is a finite disjoint union of circles.

To obtain the main result, we will use the real cusps as in Jaffee's technique.

3. Main Calculations

In this section, we look at the group $\Gamma_F(N)$ obtained by adjoining the "Fricke reflection". $F : z \to 1/N\bar{z}$ to $\Gamma_0(N)$. Thus

$$\Gamma_F(N) =< \Gamma_0(N), F >= \Gamma_0(N) \bigcup \Gamma_0(N) \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}.$$

And we will investigate the number of boundary components of $\mathbf{H}^*/\Gamma_F(N)$, where $N = n^2$. So, throughout this section N will be a squre, $N = n^2$, say. Now, we will give a definition and some lemmas necessary for our main results.

Definition 3.1. *Let* $\begin{bmatrix} a \\ b \end{bmatrix}$ *be a cusp of* $X_0(N)$. *It is called a real cusp if there exist on element* Y *of* $\Gamma_F(N)$ *such that*

$$Y \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

where $\begin{bmatrix} a \\ b \end{bmatrix}$ *denotes the orbit of* $\frac{a}{b} \in \hat{\mathbf{Q}}$ *under* $\Gamma_0(N)$.

Now, we let us investigate under what conditions a cusp point turns to a real cusp point by the reflections of $\Gamma_F(N)$. For this, it is sufficient to find the necessary and sufficient conditions for which a cusp point is fixed by the reflection $F(z) = \frac{1}{N\bar{z}}$. Because every reflection Y of $\Gamma_F(N)$ can be written as a product FE where E is an element of $\Gamma_0(N)$.

Lemma 3.2. [3, 5]

i.
$$\epsilon_i = \begin{cases} 0, & if \quad 4|N \\ \prod_{p|N}(1 + (-1/p)), & otherwise \end{cases}$$

ii.
$$\epsilon_p = \begin{cases} 0, & if \quad 9|N \\ \prod_{p|N}(1 + (-3/p)), & otherwise \end{cases}$$

iii. $\eta(N) = \sum_{d|N} \varphi((d, N/d))$, *where* φ *is Euler's function.*
Here, we understand that $\varphi(1), (\frac{-}{p})$ *is the quadratic residue symbol (in the extended sense), so that*

$$(\frac{-1}{p}) = \begin{cases} 0, & if \quad p = 2 \\ 1, & if \quad p \equiv 1 \ mod \ 4 \\ -1, & if \quad p \equiv 3 \ mod \ 4 \end{cases}$$

$$(\frac{-3}{p}) = \begin{cases} 0, & if \quad p = 3 \\ 1, & if \quad p \equiv 1 \ mod \ 3 \\ -1, & if \quad p \equiv 2 \ mod \ 3 \end{cases}$$

We shall now show that for any $\Gamma_0(N)-$ orbit in $\hat{\mathbf{Q}}$ we can choose $\frac{a}{d}$ with $d|N$ as a representative of that orbit.

Lemma 3.3. *Given an arbitrary rational number k/s with $(k,s)=1$, then there exists an element $A \in \Gamma_0(N)$ such that $A(k/s) = (k_1/s_1)$ with $s_1|N$.*

Proof. From
$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix}\begin{pmatrix} k \\ s \end{pmatrix} = \begin{pmatrix} ak+bs \\ Nck+ds \end{pmatrix},$$
we find some pairs $\{c,d\}$ for which the equation
$$Nck + ds = (N,s) \tag{3.1}$$
holds, for $(N,s)|N$, so $s_1 = (N,s)$ works. Since $(\frac{Nk}{(N,s)}, \frac{s}{(N,s)}) = 1$ there exists a pair $\{c_0, d_0\}$ so that the equation (3.1) is satisfied. Therefore the general solution of (3.1) is:
$$\begin{aligned} c &= c_0 + \tfrac{s}{(N,s)}n \\ d &= d_0 - \tfrac{Nk}{(N,s)}n, \quad \text{where } n \in \mathbf{Z}. \end{aligned} \tag{3.2}$$
Let $N = q_0^{\alpha_0} q_1^{\alpha_1} \dots q_{k_0}^{\alpha_{k_0}}$ be the prime power decomposition of N. We must show that there exists a pair $\{c_*, d_*\}$ obeying (3.2) such that
$$(Nc_*, d_*) = 1.$$
If $(d_0, N) = 1$, there is nothing to prove. If $(d_0, N) > 1$, then d_0 does have a common factor with N, q_0 say. Because of the equation (3.1), $(q_0, Nk/(N,s)) = 1$ therefore taking $n = 1$ in (3.2), we get an integer d_1 such that $q_0 \not| d_1$.

If $(d_1, N) > 1$, then d_1 has a common factor with $N,\ q_1$ say. Let $d_2 = d_1 - \frac{q_0 Nk}{(N,s)}$ then d_2 does not have q_0 and q_1 as factors.

If $(d_2, N) > 1,\ d_2$ has a common factor with $N,\ q_2$ say. Eventually we arrive at
$$d_3 = d_2 - \tfrac{q_0 q_1 Nk}{(N,s)}, \text{and so } d_3 \text{ has no } q_0, q_1, q_2 \text{ as factors}$$
$$\vdots$$
$$d_{k_0+1} = d_{k_0} - \tfrac{q_0 q_1 \dots q_{k_0-1} Nk}{(N,s)}, \text{and so } d_{k_0+1} \text{ has no } q_0 \dots q_{k_0} \text{ as factors.}$$
Hence $(d_{k_0+1}, N) = 1$. Let $d_* = d_{k_0+1}$ and the corresponding $c,\ c_*$ say, and so $(Nc_*, d_*) = 1$. This implies that there exists an element (in fact, infinitely many) $A \in \Gamma_0(N)$ such that $A\begin{pmatrix} k \\ s \end{pmatrix} = \begin{pmatrix} k_1 \\ s_1 \end{pmatrix}$ with $s_1|N$. ☐

Lemma 3.4. *Let $d_1|N$ and for some $A \in \Gamma_0(N)$. $A\begin{pmatrix} a_1 \\ d_1 \end{pmatrix} = A\begin{pmatrix} a_2 \\ d_1 \end{pmatrix}$ with $(a_1, d_1) = (a_2, d_1)$. Then $a_1 \equiv a_2$ mod t, where $t = \left(d_1, \dfrac{N}{d_1}\right)$.*

Proof. Let $A = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}$, then $A\begin{pmatrix} a_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} aa_1 + bd_1 \\ Na_1c + dd_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ d_1 \end{pmatrix}$. Therefore, $aa_1 + bd_1 = a_2$, and so $aa_1 - a_2 \equiv 0 \ mod \ d_1$
$$Na_1c + dd_1 = d_1 \quad \text{or} \quad \frac{N}{d_1}a_1c + d = 1,$$
$$aa_1 - a_2 \equiv 0 \ mod \ d_1 \text{ and } \quad aa_1 - a_2 \equiv 0 \ mod \ t.$$
From $detA$, we have $ad \equiv 1 \ mod \ t$, and from the above $d \equiv 1 \ mod \ t$ therefore $a \equiv 1 \ mod \ t$. since $aa_1 - a_2 \equiv 0 \ mod \ t$. ☐

Lemma 3.5. *Let $b|N$ and let $(a_1,b)=(a_2,b)=1$. Then $\begin{pmatrix} a_1 \\ b \end{pmatrix}$ and $\begin{pmatrix} a_2 \\ b \end{pmatrix}$ are conjugate under the action of $\Gamma_0(N)$ if and only if $a_1 \equiv a_2$ mod t, where $t=\left(b,\dfrac{N}{b}\right)$.*

Proof. If we use Lemma 3.2 and Lemma 3.4, the proof of the lemma follows easily. □

Using Lemma 3.3 and Lemma 3.5, we can see that

$$\begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{if and only if} \quad N=y^2.$$

Consequently, $\begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}$ fixes only the cusps of the form $\begin{pmatrix} x \\ n \end{pmatrix}$, where $N=n^2$.

Theorem 3.6. *Let $y|N$ and $\begin{bmatrix} x \\ y \end{bmatrix}$ be a cusp of $X_0(N)$. $F\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}$ if and only if $N=y^2$.*

Proof. If $F\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}$, there exists an element $T(z)=\frac{az+b}{cNz+d}$ of $\Gamma_0(N)$ such that $F\begin{bmatrix} x \\ y \end{bmatrix}=\Gamma\begin{bmatrix} x \\ y \end{bmatrix}$. Hence, we have that $\frac{1}{N\frac{x}{y}}=\frac{ax+by}{cNx+dy}$ in reduced form. This gives that $cNx+dy=\lambda N\frac{x}{y}$ and $ax+by=\lambda$ for $\lambda=\pm1$. From the first equation, we obtain that $y|\frac{Nx}{y}$. Taking into account that second equation and that x and y are coprime to each other, we get $y^2|N$.

Conversely, we suppose that $N=y^2$. Taking $\frac{\alpha}{\beta}\in F\begin{bmatrix} x \\ y \end{bmatrix}$, there exists an element $\frac{u}{v}$ of $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $\frac{\alpha}{\beta}=F(\frac{u}{v})$. It means that $T(\frac{x}{y})=\frac{u}{v}$ where $T\in\Gamma_0(y^2)$. Hence, we have that $\frac{ax+by}{y(cyx+d)}=\frac{u}{v}$ where $T(z)=\frac{az+b}{cy^2z+d}$. Saying $m=ax+by$ and $n=cyx+d$, we get $\frac{1}{y^2\frac{m}{yn}}=\frac{n}{ym}$. Since $(m,y)=1$, there exists an integer k such that $\frac{n}{ym}\approx\frac{k}{(ym,y^2)}=\frac{k}{y}$. As a consequence, there exists an element s of $\Gamma_0(y^2)$ such that $s(\frac{n}{ym})=\frac{k}{y}$. Writing $\frac{a_1n+b_1ym}{c_1y^2n+d_1ym}=\frac{k}{y}$, we get that $k=\lambda(a_1n+b_1ym)$ and $y=\lambda(c_1y^2n+d_1ym)$ for $\lambda=\pm1$. From the first equation, we have $y\setminus k-\lambda a_1(cyx+d)$. It means that $k\equiv\lambda a_1d$ *mod* y. since $\lambda(c_1yn+d_1m)=1$, we obtain $y\setminus 1-\lambda d_1x$. As $\lambda d_1ax\equiv 1$ *mod* y, we have $x\equiv\lambda a_1d$ *mod* y using the congruences $a_1d_1\equiv$ *mod* y and $ad\equiv$ *mod* y. Hence, it is clear that $\frac{k}{y}\approx\frac{x}{y}$. Then, we have $\frac{k}{y}\approx\frac{x}{y}$ which means that $\frac{n}{ym}\approx\frac{x}{y}$. Hence, $\frac{\alpha}{\beta}\in\begin{bmatrix} x \\ y \end{bmatrix}$. As a consequence $F\begin{bmatrix} x \\ y \end{bmatrix}\subset\begin{bmatrix} x \\ y \end{bmatrix}$. Since $F=F^{-1}$ and $F\begin{bmatrix} x \\ y \end{bmatrix}\subset\begin{bmatrix} x \\ y \end{bmatrix}$, then $\begin{bmatrix} x \\ y \end{bmatrix}\subset F\begin{bmatrix} x \\ y \end{bmatrix}$. Hence, the proof is completed. □

The following two corollaries are obtained as a consequence of above theorem:

Theorem 3.7. *Let x_1 and x_2 be an integer and let $(x_1,y)=(x_2,y)=1$, then $\left[\ \frac{x_1}{y}\ \right]=\left[\ \frac{x_2}{y}\ \right]\iff x_1\equiv x_2$ mod y.*

Theorem 3.8. *Let $N = n^2$. Then*

i) *If n is odd, then the signature of $\Gamma_F(N)$ has A boundary componenets with k cusps, boundary components with $2k$ cusps and the number of boundary components is $A/2 + \varphi(n)/2k$.*
ii) *If n is even, then the signature has A boundary componenets with one cusp and boundary components with two cusps. The number of boundary components is $A/2+\varphi(n)/2$. Here A is the number of solutions of $x^2 = -1$ and k is the order of 4 in $G = (\mathbf{Z}/n\mathbf{Z})^*$.*

Before we start proving the above theorem we give a lemma concerning our ideas.

Lemma 3.9. [10] *If $(a, n) = 1$ and the congruence $x^2 \equiv a \bmod n$ is solvable, it has exactly $2^{\sigma+\tau}$ solutions, where σ is the number of distinct odd prime divisors of n and τ is $0,\ 1$ and 2 according as $4 \not| n,\ 2^2||n$ or $8|n$.*

From this lemma we know the exact value of A in Theorem 3.8.
Proof of Theorem 3.4

Any reflection T in $\Gamma_F(N)$ is of the form $\begin{pmatrix} aN & b \\ cN & -aN \end{pmatrix}$ of determinant $-N$ and fixed points (real cusps) of the reflection are $(-1 + an)/cn$ and $(1 + an)/cn$. Since any reflection in $\Gamma_F(N)$ fixes real cusps, every boundary conponent has at least one ∞.

i) Let n be odd. Then $\det T = a^2n^2 + bc = 1$. So $bc = (1-an)(1+an)$. Firstly suppose c is odd. Write $c = c_1c_2$ such that $c_1|(-1 + an)$ and $c_2|(1 + an)$ with $(c_2, c_1) = 1$. Therefore we get $\frac{\frac{-1+an}{c_1}}{c_2 n}$ and $\frac{\frac{1+an}{c_2}}{c_1 n}$. Since $(c_2n, n^2) = (c_1n, n^2) = n$, $\frac{\frac{-1+an}{c_1}}{c_2 n} \equiv \frac{x}{n}$and $\frac{\frac{1+an}{c_2}}{c_1 n} \equiv \frac{v}{n}$, where $x \equiv c_2\frac{-1+an}{c_1} \bmod n$ and $y \equiv c_1\frac{1+an}{c_2} \bmod n$ by Lemma 3.3. So we get $xy \equiv -1 \bmod n$. Hence
$$\begin{pmatrix} x \\ n \end{pmatrix} \text{ and } \begin{pmatrix} -x^{-1} \\ n \end{pmatrix} \tag{3.3}$$
are on the same circle (boundary component). Secondly suppose that c is even. Then write $c = 2c_1c_2,\ c_2$ being odd, such that $(c_1, c_2) = 1$ and $c_1|(-1 + an),\ 2c_2|(1 + an)$. So we get $\frac{\frac{-1+an}{2c_1}}{c_2 n} \equiv \frac{x}{n}$, $\frac{\frac{1+an}{2c_2}}{c_1 n} \equiv \frac{v}{n}$, where $x \equiv c_2\frac{-1+an}{2c_1} \bmod n$ and $y \equiv c_1\frac{1+an}{2c_2} \bmod n$. Therefore $4xy \equiv -1 \bmod n$. Hence
$$\begin{pmatrix} x \\ n \end{pmatrix} \text{ and } \begin{pmatrix} -(4x)^{-1} \\ n \end{pmatrix} \tag{3.4}$$
are on the same circle (boundary component). Consequently, combining (3.3) and (3.4) we get

$$\begin{pmatrix} -x^{-1} \\ n \end{pmatrix}, \begin{pmatrix} x \\ n \end{pmatrix}, \begin{pmatrix} -(4x)^{-1} \\ n \end{pmatrix}, \begin{pmatrix} 4x \\ n \end{pmatrix}, \begin{pmatrix} -(16x)^{-1} \\ n \end{pmatrix}, \begin{pmatrix} 16x \\ n \end{pmatrix}, \ldots,$$
$$\begin{pmatrix} -(4^{k-1}x)^{-1} \\ n \end{pmatrix}, \begin{pmatrix} 4^{k-1}x \\ n \end{pmatrix}, \tag{3.5}$$

where k is the order of 4 in $G = (\mathbf{Z}/n\mathbf{Z})^*$. The length of (3.5) is $2k$ except for the following two cases.

In the first case. If $x^2 \equiv -1\ mod\ n$ (or $x^2 = -1$ in G) then $-(4^{m_1}x)^{-1}/n \equiv 4^{m_2}x/n$ by Lemma 3.5, where $m_1,\ m2 \leq k$ and $m_1 + m_2 \equiv 0\ mod\ k$. So in this case (3.5) has length k.
In the second case. If $4x^2 \equiv -1$ in G, then $-(4^{m_1}x)/n \equiv 4^{k-m_1+1}x/n$ and the length of (3.5) in this case is k as well.
Hence we now show that there are really A boundary components with k cusps.To do this, we now deal with the above two cases:
Since n is odd $A = 2^r$ by Lemma 3.9, where r is the number of distinct primes of n.
Let x_i be odd solutions of the congruence $x^2 \equiv -1\ mod\ n$, where $i = 1, 2, \ldots, 2^{r-1}$. Let $y_i = -x_i$ in G and therefore the y_i are the even solutions of $x^2 \equiv -1\ mod\ n$. We now list all solutions of the congruence $x^2 \equiv -1\ mod\ n$ in the first two columns of the following table.

Table 1.

odd	even		
x_1	y_1	y_1^1	z_1
x_2	y_2	y_2^1	z_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	y	y^1	z
$x_{2^{r-1}}$	$y_{2^{r-1}}$	$y_{2^{r-1}}^1$	$z_{2^{r-1}}$

where $y_i^1 = y_i/2$ and $z_i = -y_i^1/2$ in G, $i = 1, \ldots, 2^{r-1}$. From the last two columns of Table1 we conclude that

$$4(y_i^1)^2 \equiv 4(z_i)^2 \equiv -1\ mod\ n.$$

Then we show that every component (circle) contains only two cusps whose numerators are in the above Table1. Suppose now k is even. Assume that there is at least one component which contains three different real cusps with numerators $x_1,\ y_1,\ y_1^1$, say. Therefore $\begin{pmatrix} x_1 \\ n \end{pmatrix}$, $\begin{pmatrix} y_1 \\ n \end{pmatrix}$ and $\begin{pmatrix} y_1^1 \\ n \end{pmatrix}$ are on the same circle (so three ∞ on the same boundary component). Using (3.5) we get $4^m x_1 \equiv y_1\ mod\ n$, then $-4^m \equiv x_1 y_1\ mod\ n$, since $x_1^2 \equiv -1\ mod\ n$. $4^m x_1 y_1 \equiv -1\ mod\ n$ since $y_1^2 \equiv -1\ mod\ n$. Hence $4^{2m} \equiv 1\ mod\ n$ and therefore $k = 2m$. And $4^s x_1 \equiv y_1^1\ mod\ n$, where $s < k$ and $x_1^2 \equiv 4y_1^2\ \equiv -1\ mod\ n$. So $4^{s+1}x_1 \equiv 4y_1^1\ mod\ n$ then $4^{s+1}x_1 y_1^1 \equiv -1\ mod\ n$ and $-4^s \equiv x_1 y_1^1\ mod\ n$. Hence $4^{2s+1} \equiv 1\ mod\ n$ implies that $k = 2s+1$ which is a contradiction to k being even.
The other possibilities being three cusps with numerators in Table1 on the same circle can be treated similarly. Therefore every circle contains exactly two real cusps whose numerators are in Table1. In Table1 there are 2.2^r points, so there are $A = 2^r$ circles with k real cusps. Consequently 2^r boundary components with k ∞'s each. Now, assume that k is odd. Then, with a similar argument as above, we conclude that every circle has exactly two real cusps with numerators in Table1. In this case if $x^2 = -1$ in G then $\begin{pmatrix} x \\ n \end{pmatrix}$ and $\begin{pmatrix} 4^{(k-1)/2}x \\ n \end{pmatrix}$ which are different are on the same circle.

Consequently by 3.5 we conclude that the number of boundary components is $A/2 + \varphi(n)/2k$, where k is as above, as required.

ii) Suppose now n is even. Then the fixed cusps of the reflection

$$T = \begin{pmatrix} an^2 & b \\ cn^2 & -an^2 \end{pmatrix}$$

of determinant $-n^2$ are $(-1+an)/cn$ and $(1+an)/cn$. Here c is always odd. $detT = a^2n^2 + bc = 1$, so $bc = (1-an)(1+an)$. Write $c = c_1c_2$ so that $c_1|(1-an)$ and $c_2|(1+an)$ with (c_1, c_2)=1. As we have done in i) if $x/n \equiv (-1+an)/cn$ and $y/n \equiv (1+an)/cn$ by Lemma 3.5 we get $xy \equiv -1 \ mod \ n$. So

$$\begin{pmatrix} x \\ n \end{pmatrix} \text{ and } \begin{pmatrix} -x^{-1} \\ n \end{pmatrix} \tag{3.6}$$

are the only real cusps on a circle. But if $x^2 \equiv -1 \ mod \ n$ then $x/n \equiv -x^{-1}/n$. Therefore by (3.6) we have A boundary components with one ∞ and boundary components with 2 ∞'s. Consequently the number of boundary components is just $A/2 + \varphi(n)/2$, as promised.

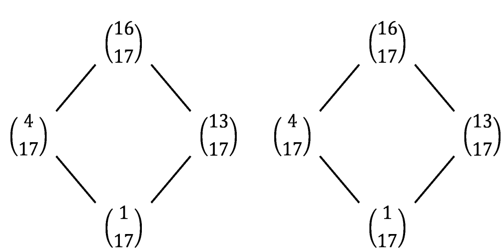
Examples

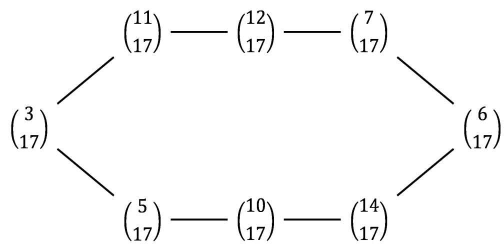
1. Let $N = 17^2$. By Theorem 3.8 the number of real cusps is $\varphi(n) = \varphi(17) = 16$ and since $\sigma = 1, \ \tau = 0, \ A = 2^{\sigma+\tau} = 2$. The smallest integer k satisfying the congruence $4^k \equiv 1 \ mod \ 17$ is 4. So the solutions of the congruence $x^2 \equiv -1 \ mod \ 17$ are in the first two columns of the following table in accordance with the Table1.

Table 2.

x_i	y_i	$u_i = y_i/2$	$v_i = -u_1$
$x_1 = 4$	$y_1 = 13$	$u_1 = 15$	$v_1 = 2$
$x_2 = 13$	$y_2 = 4$	$u_2 = 2$	$v_2 = 15$

Using (3.5), we get circles of the real cusps as follows:





For the signature $\sigma(\Gamma_F(17^2))$; each of the $A = 2$ boundary components have $k = 4$ infinite link periods and the other $\frac{\varphi(n)}{2k} - \frac{A}{2} = \frac{16}{8} - \frac{2}{2} = 1$ boundary components have $2k = 8$ infinite link periods.

2. Let $N = 34^2$. Then $n = 34 = 2.17$ and $\varphi(34) = 16$. The number A of solutions of the congruence $x^2 \equiv -1 \ mod \ 34$ is 2. The solutions are 13 and 21. Therefore using (3.6) we get the circles of the real cusps as follows:

$$\begin{array}{ccc}
\binom{1}{34} \leftrightarrow \binom{33}{34} & \binom{3}{34} \leftrightarrow \binom{11}{34} & \binom{5}{34} \leftrightarrow \binom{27}{34} \\
\binom{7}{34} \leftrightarrow \binom{29}{34} & \binom{15}{34} \leftrightarrow \binom{9}{34} & \binom{19}{34} \leftrightarrow \binom{25}{34} \\
\binom{21}{34} \circlearrowright & \binom{23}{34} \leftrightarrow \binom{31}{34} & \circlearrowleft \binom{13}{34}
\end{array}$$

When n is even, for the signature $\sigma(\Gamma_F(34^2))$; each of the $A = 2$ boundary components have 1 infinite link period and the other $\frac{\varphi(34)}{2} - \frac{2}{2} = \frac{16}{2} - \frac{2}{2} = 7$ boundary components have 2 infinite link periods.

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