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SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this article, we will first describe a new generalized double fractional integrals that generalize some important fractional integrals such as Riemann-Liouville fractional integrals on two coordinates, $k-$Riemann-Liouville fractional integrals on two coordinates, double conformable fractional integrals, etc with some special choices. Later, using this generalized fractional integral we just described, we will obtain new inequalities similar to the right side of the Hermite-Hadamard inequality.

1. Introduction

Fractional calculus and applications have application areas in many different fields such as physics, chemistry and engineering as well as mathematics. The application of arithmetic carried out in classical analysis in fractional analysis is very important in terms of obtaining more realistic results in the solution of many problems. Many real dynamical systems are better characterized by using non-integer order dynamic models based on fractional computation. While integer orders are a model that is not suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined, enables us to obtain more realistic approaches. This subject has been studied by many scientists in terms of its widespread use, see ([2]-[5], [7], [8], [10]-[12], [14], [15], [17]-[20],[24]). One of the most important applications of the fractional integrals is the Hermite-Hadamard integral inequality, see ([1], [4], [6], [7], [9], [13], [16], [17], [20]-[23]).

Hermite sent a letter in which he mentioned an inequality obtained by showing convex functions to Mathesis magazine in 1881 and this study was published with the file extension "Mathesis 3 p.82, 1883". Unfortunately, Hermite's main work is often mentioned anonymously. Later, similar results were mentioned in a study conducted by Hadamard [9] in 1893. This important inequality has taken its place in the literature as the Hermite-Hadamard inequality:

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$$g\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x)\,dx \leq \frac{g(a_1)+g(a_2)}{2}$$

where $f : I \to \mathbb{R}$ is a convex mapping defined on the interval $I \subseteq \mathbb{R}$ and $a, b \in I$, with $a < b$.

In this paper, we get some inequalities similar to the Hermite-Hadamard inequality on a rectangle from the plane $\mathbb{R}^2$.

For our work to be done in two-dimensional space, we need the following description:

Definition 1.1. *Let $\Delta =: [a_1, a_2] \times [b_1, b_2]$ in $\mathbb{R}^2$ with $a_1 < a_2$ and $b_1 < b_2$. A mapping $g : \Delta \to \mathbb{R}$ is said to be convex in the bidimensional interval Δ, if the following inequality holds:*

$$\begin{aligned} &g(tx+(1-t)y, sz+(1-s)w) \\ &\leq tsg(x,z)+s(1-t)g(y,z)+t(1-s)g(a_1,w)+(1-t)(1-s)g(y,w) \end{aligned} \tag{1.1}$$

for all $(x, y), (z, w) \in \Delta$ and $t, s \in [0, 1]$.

If the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex for all $x \in [a, b]$ and $y \in [c, d]$, then $f : \Delta \to \mathbb{R}$ is called convex in coordinates. However, the opposite is not correct (see, [6]).

Structurally similar inequality to Hermite-Hadamard inequality was obtained by Dragomir for co-ordinated convex functions in [6].

Theorem 1.2. *Suppose that $g : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then the below sharp inequalities hold*:

$$\begin{aligned} &g\left(\tfrac{a_1+a_2}{2}, \tfrac{b_1+b_2}{2}\right) \\ &\leq \tfrac{1}{2}\left[\tfrac{1}{a_2-a_1}\int_{a_1}^{a_2} g\left(x, \tfrac{b_1+b_2}{2}\right)dx + \tfrac{1}{b_2-b_1}\int_{b_1}^{b_2} g\left(\tfrac{a_1+a_2}{2}, y\right)dy\right] \\ &\leq \tfrac{1}{(a_2-a_1)(b_2-b_1)}\int_{a_1}^{a_2}\int_{b_1}^{b_2} g(x,y)\,dydx \\ &\leq \frac{1}{4}\left[\tfrac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x,b_1)\,dx + \tfrac{1}{a_2-a_1}\int_{a_1}^{a_2} g(x,b_2)\,dx\right. \\ &\left.+\tfrac{1}{b_2-b_1}\int_{b_1}^{b_2} g(a_1,y)\,dy + \tfrac{1}{b_2-b_1}\int_{b_1}^{b_2} g(a_2,y)\,dy\right] \\ &\leq \tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}. \end{aligned} \tag{1.2}$$

The proof of a special version of Theorem 1 was shown by Sarikaya and Yaldiz in [21].

Recently there have been many important results related with Hermite-Hadamard inequality for co-ordinated convex functions (see, [6], [13], [20], [23]).

Under the same hypotheses, part of the expression in Theorem 1 was recovered by Sarıkaya in [20] with the help of two variable fractional integrals:

Theorem 1.3. *Let $g: \triangle \subset \mathbb{R}^2 \to \mathbb{R}$ be co-ordinated convex on $\triangle := [a_1, a_2] \times [b_1, b_2]$ in $\mathbb{R}^2$ with $0 \leq a_1 < a_2$, $0 \leq b_1 < b_2$ and $g \in L_1(\triangle)$. Then one has the inequalities:*

$$\begin{aligned} & g\left(\tfrac{a_1+a_2}{2}, \tfrac{b_1+b_2}{2}\right) \\ & \leq \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(a_2-a_1)^{\alpha}(b_2-b_1)^{\beta}} \left[J^{\alpha,\beta}_{a_1^+,b_1^+} g(a_2,b_2) + J^{\alpha,\beta}_{a_1^+,b_2^-} g(a_2,b_1) \right. \\ & \left. + J^{\alpha,\beta}_{a_2^-,b_1^+} g(a_1,b_2) + J^{\alpha,\beta}_{a_2^-,b_2^-}(a_1,b_1) \right] \\ & \leq \tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}. \end{aligned} \tag{1.3}$$

In [22], Sarıkaya et al. described the new left-sided and right-sided generalized fractional integrals as the following:

$$ {}_{a_1^+}I_{\varphi} g(x) = \int_{a_1}^{x} \frac{\varphi(x-t)}{x-t} g(t)\, dt, \ x > a_1 \tag{1.4}$$

$$ {}_{a_2^-}I_{\varphi} g(x) = \int_{x}^{a_2} \frac{\varphi(t-x)}{t-x} g(t)\, dt, \ x < a_2, \tag{1.5}$$

respectively, where $\varphi : [0,\infty) \to [0,\infty)$ is a function which satisfies $\int_0^1 \frac{\varphi(t)}{t} dt < \infty$.

Furthermore, they noticed that these generalized fractional integrals may contain some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integral, conformable fractional integral, etc., with some special choices.

Inspired by this definition, we give the following definitions:

Definition 1.4. *Let $g \in L_1([a_1,a_2]\times[b_1,b_2])$. The Generalized Riemann–Liouville integrals ${}_{a_1^+,b_1^+}I_{\varphi,\psi,a_1^+,b_2^-}$ $I_{\varphi,\psi,a_2^-,b_1^+}$ $I_{\varphi,\psi,a_2^-,b_2^-}$ $I_{\varphi,\psi}$ are defined by*

$$ {}_{a_1^+,b_1^+}I_{\varphi,\psi} g(x,y) = \int_{a_1}^{x}\int_{b_1}^{y} \frac{\varphi(x-t)}{x-t}\frac{\psi(y-s)}{y-s} g(t,s)\, dsdt, \ x > a_1, \ y > b_1, \tag{1.6}$$

$$ {}_{a_1^+,b_2^-}I_{\varphi,\psi} g(x,y) = \int_{a_1}^{x}\int_{y}^{b_2} \frac{\varphi(x-t)}{x-t}\frac{\psi(s-y)}{s-y} g(t,s)\, dsdt, \ x > a_1, \ y < b_2, \tag{1.7}$$

$$ {}_{a_2^-,b_1^+}I_{\varphi,\psi} g(x,y) = \int_{x}^{a_2}\int_{b_1}^{y} \frac{\varphi(t-x)}{t-x}\frac{\psi(y-s)}{y-s} g(t,s)\, dsdt, \ x < a_2, \ y > b_1, \tag{1.8}$$

and

$$ {}_{a_2^-,b_2^-}I_{\varphi,\psi} g(x,y) = \int_{x}^{a_2}\int_{y}^{b_2} \frac{\varphi(t-x)}{t-x}\frac{\psi(s-y)}{s-y} g(t,s)\, dsdt, \ x < a_2, \ y < b_2, \tag{1.9}$$

where $\varphi, \psi : [0,\infty) \to [0,\infty)$ functions which satisfy $\int_0^1 \frac{\varphi(t)}{t} dt < \infty$ and $\int_0^1 \frac{\psi(s)}{s} ds < \infty$, respectively.

Using the above definitions, the following can also be written

$$ {}_{a_1^+}I_{\varphi} g\left(x, \frac{b_1+b_2}{2}\right) = \int_{a_1}^{x} \frac{\varphi(x-t)}{x-t} g\left(t, \frac{b_1+b_2}{2}\right) dt, \ x > a_1,$$

$$ {}_{a_2^-}I_{\varphi} g\left(x, \frac{b_1+b_2}{2}\right) = \int_{x}^{a_2} \frac{\varphi(t-x)}{t-x} g\left(t, \frac{b_1+b_2}{2}\right) dt, \ x < a_2,$$

$$ {}_{b_1^+}I_{\psi} g\left(\frac{a_1+a_2}{2}, y\right) = \int_{b_1}^{y} \frac{\psi(y-s)}{y-s} g\left(\frac{a_1+a_2}{2}, s\right) ds, \ y > b_1,$$

and

$$ {}_{b_2^-}I_{\psi}g\left(\frac{a_1+a_2}{2},y\right)=\int_y^{b_2}\frac{\psi\left(s-y\right)}{s-y}g\left(\frac{a_1+a_2}{2},s\right)ds,\ y<b_2. $$

In this definition, known fractional integrals can be obtained by some special choices. For example;

i) If we take $\varphi\left(t\right)=t$ and $\psi\left(s\right)=s$, then the operators (1.6), (1.7), (1.8) and (1.9) transform into the the Riemann integrals on two coordinates respectively as the following

$$ J^{\alpha,\beta}_{a_1^+,b_1^+}g\left(x,y\right)=\int_{a_1}^{x}\int_{b_1}^{y}g\left(t,s\right)dsdt,\ x>a_1,\ y>b_1, $$

$$ J^{\alpha,\beta}_{a_1^+,b_2^-}g\left(x,y\right)=\int_{a_1}^{x}\int_{y}^{b_2}g\left(t,s\right)dsdt,\ x>a_1,\ y<b_2, $$

$$ J^{\alpha,\beta}_{a_2^-,b_1^+}g\left(x,y\right)=\int_{x}^{a_2}\int_{b_1}^{y}g\left(t,s\right)dsdt,\ x<a_2,\ y>b_1, $$

and

$$ J^{\alpha,\beta}_{a_2^-,b_2^-}g\left(x,y\right)=\int_{x}^{a_2}\int_{y}^{b_2}g\left(t,s\right)dsdt,\ x<a_2,\ y<b_2. $$

ii) If we take $\varphi\left(t\right)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, $\psi\left(s\right)=\frac{s^{\beta}}{\Gamma(\beta)}$, then for $\alpha,\beta>0$ the operators (1.6), (1.7), (1.8) and (1.9) transform into the Riemann-Liouville integrals on two coordinates ([20]) respectively as the following

$$ \begin{aligned} &J^{\alpha,\beta}_{a_1^+,b_1^+}g\left(x,y\right)\\ &=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a_1}^{x}\int_{b_1}^{y}\left(x-t\right)^{\alpha-1}\left(y-s\right)^{\beta-1}g\left(t,s\right)dsdt,\ x>a_1,\ y>b_1, \end{aligned} $$

$$ \begin{aligned} &J^{\alpha,\beta}_{a_1^+,b_2^-}g\left(x,y\right)\\ &=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a_1}^{x}\int_{y}^{b_2}\left(x-t\right)^{\alpha-1}\left(s-y\right)^{\beta-1}g\left(t,s\right)dsdt,\ x>a_1,\ y<b_2, \end{aligned} $$

$$ \begin{aligned} &J^{\alpha,\beta}_{a_2^-,b_1^+}g\left(x,y\right)\\ &=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{a_2}\int_{b_1}^{y}\left(t-x\right)^{\alpha-1}\left(y-s\right)^{\beta-1}g\left(t,s\right)dsdt,\ x<a_2,\ y>b_1, \end{aligned} $$

and

$$ \begin{aligned} &J^{\alpha,\beta}_{a_2^-,b_2^-}g\left(x,y\right)\\ &=\tfrac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{a_2}\int_{y}^{b_2}\left(t-x\right)^{\alpha-1}\left(s-y\right)^{\beta-1}g\left(t,s\right)dsdt,\ x<a_2,\ y<b_2, \end{aligned} $$

along with Γ called Gamma function.

iii) If we take $\varphi\left(t\right)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi\left(s\right)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, for $\alpha,\beta,k>0$ then the operators (1.6), (1.7), (1.8) and (1.9) transform into the Riemann-Liouville $k-$fractional integrals on two coordinates ([7]) respectively as the following

$$ \begin{aligned} &{}_kI^{\alpha,\beta}_{a_1^+,b_1^+}g\left(x,y\right)\\ &=\tfrac{1}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\int_{a_1}^{x}\int_{b_1}^{y}\left(x-t\right)^{\frac{\alpha}{k}-1}\left(y-s\right)^{\frac{\beta}{k}-1}g\left(t,s\right)dsdt,\ x>a_1,\ y>b_1, \end{aligned} $$

$$
\begin{aligned}
&{}_kI^{\alpha,\beta}_{a_1^+,b_2^-}g(x,y)\\
&=\tfrac{1}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\int_{a_1}^{x}\int_{y}^{b_2}(x-t)^{\frac{\alpha}{k}-1}(s-y)^{\frac{\beta}{k}-1}g(t,s)\,dsdt,\ x>a_1,\ y<b_2,\\
&{}_kI^{\alpha,\beta}_{a_2^-,b_1^+}g(x,y)\\
&=\tfrac{1}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\int_{x}^{a_2}\int_{b_1}^{y}(t-x)^{\frac{\alpha}{k}-1}(y-s)^{\frac{\beta}{k}-1}g(t,s)\,dsdt,\ x<a_2,\ y>b_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_kI^{\alpha,\beta}_{a_2^-,b_2^-}g(x,y)\\
&=\tfrac{1}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\int_{x}^{a_2}\int_{y}^{b_2}(t-x)^{\frac{\alpha}{k}-1}(s-y)^{\frac{\beta}{k}-1}g(t,s)\,dsdt,\ x<a_2,\ y<b_2,
\end{aligned}
$$

along with Γ_k called $k-$Gamma function.

iv) If we take $\varphi(t)=t(a_2-t)^{\alpha-1}$ and $\psi(s)=s(b_2-s)^{\alpha-1}$, for $\alpha\in(0,1]$ then the operators (1.6), (1.7), (1.8) and (1.9) transform into double conformable fractional integral ([18]) respectively as the following

$$
\begin{aligned}
&\int_{a_1}^{a_2}\int_{b_1}^{b_2}t^{\alpha-1}s^{\alpha-1}g(t,s)\,dsdt\\
&=\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(t,s)\,d_\alpha s d_\alpha t,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{a_1}^{a_2}\int_{b_1}^{b_2}t^{\alpha-1}(b_2+b_1-s)^{\alpha-1}g(t,s)\,dsdt\\
&=\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(t,b_2+b_1-v)\,d_\alpha v d_\alpha t,\\
&\int_{a_1}^{a_2}\int_{b_1}^{b_2}(a_2+a_1-t)^{\alpha-1}s^{\alpha-1}g(t,s)\,dsdt\\
&=\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(a_2+a_1-u,s)\,d_\alpha s d_\alpha u,\\
&\int_{a_1}^{a_2}\int_{b_1}^{b_2}(a_2+a_1-t)^{\alpha-1}(b_2+b_1-s)^{\alpha-1}g(t,s)\,dsdt\\
&=\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(a_2+a_1-u,b_2+b_1-v)\,d_\alpha v d_\alpha u.
\end{aligned}
$$

Similarly, with some special choices, many known fractional integrals can be obtained.

Meanwhile, Sarikaya et al. [22] gave the following theorem which is about Herimite-Hadamard inequality for the generalized fractional integral operators.

Theorem 1.5. *Let $g:[a_1,a_2]\to\mathbb{R}$ be convex function on $[a_1,a_2]$ with $a_1<a_2$, then the following inequalities for the generalized fractional integral hold:*

$$
g\left(\frac{a_1+a_2}{2}\right)\leq\frac{1}{2\Lambda(1)}\left[{}_{a_1^+}I_{\varphi}g(a_2)+{}_{a_2^-}I_{\varphi}g(a_1)\right]\leq\frac{g(a_1)+g(a_2)}{2}\tag{1.10}
$$

where $\Lambda(x)$ defined by

$$
\Lambda(x)=\int_0^x\frac{\varphi((a_2-a_1)t)}{t}dt.
$$

The purpose of this paper primarily is to obtain some new Hermite-Hadamard type inequalities by using the generalized Riemann-Liouville fractional integral with two variables we just defined. Another aim is to obtain new inequalities related to the right-hand part of the Hermite-Hadamard type inequality by using the fractional integral with two variables that we just described.

2. Hermite-Hadamard type inequalities for Generalized double Fractional Integrals

In this section, it is aimed to give some new Hermite-Hadamard type inequalities via the generalization of Riemann-Liouville fractional integrals on two coordinates for coordinated convex functions.

Theorem 2.1. *Let $g: \triangle \subset \mathbb{R}^2 \to \mathbb{R}$ be co-ordinated convex on $\triangle := [a_1, a_2]\times[b_1, b_2]$ in $\mathbb{R}^2$ with $a_1 < a_2$, $b_1 < b_2$ and $g \in L_1(\triangle)$. In this case, the following inequality is achieved:*

$$\begin{aligned} & g\left(\tfrac{a_1+a_2}{2}, \tfrac{b_1+b_2}{2}\right) \\ & \leq \tfrac{1}{4\Lambda(1)\Psi(1)} \left[{}_{a_1^+,b_1^+}I_{\varphi,\psi} g(a_2, b_2) + {}_{a_1^+,b_2^-}I_{\varphi,\psi} g(a_2, b_1) \right. \\ & \left. + {}_{a_2^-,b_1^+}I_{\varphi,\psi} g(a_1, b_2) + {}_{a_2^-,b_2^-}I_{\varphi,\psi} g(a_1, b_1) \right] \\ & \leq \tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4} \end{aligned} \tag{2.1}$$

where $\varphi, \psi : [0, \infty) \to [0, \infty)$ are defined in Definition 1.4 and $\Psi(x)$ is defined by

$$\Psi(x) = \int_0^x \frac{\psi((b_2 - b_1)s)}{s} ds.$$

Proof. In inequality (1.1), $x = ta_1 + (1-t)a_2, y = (1-t)a_1 + ta_2, u = sb_1 + (1-s)b_2$, $w = (1-s)b_1 + sb_2$ with variable substitution and by choosing $t = s = \frac{1}{2}$, we find that

$$\begin{aligned} & g\left(\tfrac{a_1+a_2}{2}, \tfrac{b_1+b_2}{2}\right) \\ & \leq \tfrac{1}{4}\left[g(ta_1 + (1-t)a_2, sb_1 + (1-s)b_2)\right. \\ & + g(ta_1 + (1-t)a_2, (1-s)b_1 + sb_2) \\ & + g((1-t)a_1 + ta_2, (1-s)b_1 + sb_2) \\ & \left. + g((1-t)a_1 + ta_2, (1-s)b_1 + sb_2)\right]. \end{aligned} \tag{2.2}$$

Thus, multiplying both sides of (2.2) by $\frac{\varphi((a_2-a_1)t)}{t}\frac{\psi((b_2-b_1)s)}{s}$, then by integrating with respect to (t, s) on $[0, 1] \times [0, 1]$, we obtain

$$\begin{aligned} & \Lambda(1)\Psi(1) g\left(\tfrac{a_1+a_2}{2}, \tfrac{b_1+b_2}{2}\right) \\ & \leq \frac{1}{4}\left\{\int_0^1\int_0^1 \tfrac{\varphi((a_2-a_1)t)}{t}\tfrac{\psi((b_2-b_1)s)}{s}\right. \\ & \times [g(ta_1 + (1-t)a_2, sb_1 + (1-s)b_2)] \\ & + g(ta_1 + (1-t)a_2, (1-s)b_1 + sb_2)]\, dsdt \\ & + \int_0^1\int_0^1 \tfrac{\varphi((a_2-a_1)t)}{t}\tfrac{\psi((b_2-b_1)s)}{s} \\ & \times [g((1-t)a_1 + ta_2, sb_1 + (1-s)b_2) \\ & \left. + g((1-t)a_1 + ta_2, (1-s)b_1 + sb_2)]\, dsdt\right\}. \end{aligned}$$

Using the change of the variable, we get

$$\begin{aligned}
&\Lambda(1)\Psi(1)g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right)\\
&\leq \tfrac{1}{4}\left\{\int_{a_1}^{a_2}\int_{b_1}^{b_2}\tfrac{\varphi(a_2-x)}{a_2-x}\tfrac{\psi(b_2-y)}{b_2-y}g\left(x,y\right)dydx\right.\\
&+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\tfrac{\varphi(a_2-x)}{a_2-x}\tfrac{\psi(y-b_1)}{y-b_1}g\left(x,y\right)dydx\\
&+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\tfrac{\varphi(x-a_1)}{x-a_1}\tfrac{\psi(b_2-y)}{b_2-y}g\left(x,y\right)dydx\\
&\left.+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\tfrac{\varphi(x-a_1)}{x-a_1}\tfrac{\psi(y-b_1)}{y-b_1}g\left(x,y\right)dydx\right\}\\
&=\tfrac{1}{4}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}I_{\varphi,\psi}g\left(a_2,b_1\right)\right.\\
&\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}I_{\varphi,\psi}g\left(a_1,b_1\right)\right].
\end{aligned}$$

Therefore, the first part of the inequality is obtained. To get the last part of the inequality of 2.1, we will use the function g being co-ordinated convex on $\triangle$, then function g satisfies the inequality (1.1)

$$\begin{aligned}
&g\left(ta_1+(1-t)a_2,\ sb_1+(1-s)b_2\right)\\
&\leq tsg\left(a_1,b_1\right)+s\left(1-t\right)g\left(a_2,b_1\right)\\
&+t\left(1-s\right)g\left(a_1,b_2\right)+\left(1-t\right)\left(1-s\right)g\left(a_2,b_2\right),
\end{aligned}$$

$$\begin{aligned}
&g\left(ta_1+(1-t)a_2,(1-s)b_1+sb_2\right)\\
&\leq t\left(1-s\right)g\left(a_1,b_1\right)+\left(1-t\right)\left(1-s\right)g\left(a_2,b_1\right)\\
&+tsg\left(a_1,b_2\right)+\left(1-t\right)sg\left(a_2,b_2\right),
\end{aligned}$$

$$\begin{aligned}
&g\left(\left(1-t\right)a_1+ta_2,sb_1+(1-s)b_2\right)\\
&\leq \left(1-t\right)sg\left(a_1,b_1\right)+stg\left(a_2,b_1\right)\\
&+\left(1-t\right)\left(1-s\right)g\left(a_1,b_2\right)+t\left(1-s\right)g\left(a_2,b_2\right)
\end{aligned}$$

and

$$\begin{aligned}
&g\left(\left(1-t\right)a_1+ta_2,(1-s)b_1+sb_2\right)\\
&\leq \left(1-t\right)\left(1-s\right)g\left(a_1,b_1\right)+t\left(1-s\right)g\left(a_2,b_1\right)\\
&+\left(1-t\right)sg\left(a_1,b_2\right)+tsg\left(a_2,b_2\right).
\end{aligned}$$

By adding the four inequalities above, we get

$$\begin{aligned}
&g\left(ta_1+(1-t)a_2,sb_1+(1-s)b_2\right)\\
&+g\left(ta_1+(1-t)a_2,(1-s)b_1+sb_2\right)\\
&+g\left(\left(1-t\right)a_1+ta_2,sb_1+(1-s)b_2\right)\\
&+g\left(\left(1-t\right)a_1+ta_2,(1-s)b_1+sb_2\right)\\
&\leq g\left(a_1,b_1\right)+g\left(a_2,b_1\right)+g\left(a_1,b_2\right)+g\left(a_2,b_2\right).
\end{aligned}\tag{2.3}$$

Then, multiplying both sides of (2.3) by $\frac{\varphi((a_2-a_1)t)}{t}\frac{\psi((b_2-b_1)s)}{s}$ and integrating with according to (t,s) over $[0,1]\times[0,1]$, we get

$$\begin{aligned}
&\int_0^1\int_0^1 \tfrac{\varphi((a_2-a_1)t)}{t}\tfrac{\psi((b_2-b_1)s)}{s}\\
&\times\left[g\left(ta_1+(1-t)a_2, sb_1+(1-s)b_2\right)+g\left(ta_1+(1-t)a_2,(1-s)b_1+sb_2\right)\right.\\
&\left.+g\left((1-t)a_1+ta_2, sb_1+(1-s)b_2\right)+g\left((1-t)a_1+ta_2,(1-s)b_1+sb_2\right)\right]dsdt\\
&\leq\int_0^1\int_0^1 \tfrac{\varphi((a_2-a_1)t)}{t}\tfrac{\psi((b_2-b_1)s)}{s}\\
&\times\left[g\left(a_1,b_1\right)+g\left(a_2,b_1\right)+g\left(a_1,b_2\right)+g\left(a_2,b_2\right)\right]dsdt.
\end{aligned}$$

Here, using the change of the variable we have

$$\begin{aligned}
&\tfrac{1}{4}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}I_{\varphi,\psi}g\left(a_2,b_1\right)\right.\\
&\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}I_{\varphi,\psi}g\left(a_1,b_1\right)\right]\\
&\leq\Lambda(1)\Psi(1)\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}.
\end{aligned}$$

Thus, the last part of the inequality is obtained. □

Remark. *Under the hypotheses of Theorem 2.1, when $\varphi(t)=t$ and $\psi(s)=s$ are taken, (2.1) and (1.2) become the same expression;*

$$\begin{aligned}
g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right)&\leq\tfrac{1}{(a_2-a_1)(b_2-b_1)}\int_{a_1}^{a_2}\int_{b_1}^{b_2}g\left(x,y\right)dydx\\
&\leq\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}.
\end{aligned}$$

Remark. *Under the hypotheses of Theorem 2.1, when $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ are taken, (2.1) and (1.3) become the same expression;*

$$\begin{aligned}
&g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right)\\
&\leq\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(a_2-a_1)^\alpha(b_2-b_1)^\beta}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}I_{\varphi,\psi}g\left(a_2,b_1\right)\right.\\
&\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}I_{\varphi,\psi}g\left(a_1,b_1\right)\right]\\
&\leq\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}.
\end{aligned}$$

Remark. *Under the hypotheses of Theorem 2.1, when $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ are taken, Theorem 2.1 and Theorem 3.5 in* [7] *become the same expression.*

Corollary 2.2. *Under the hypotheses of Theorem 2.1, when $\varphi(t)=t\left(a_2-t\right)^{\alpha-1}$ and $\psi(s)=s\left(b_2-s\right)^{\alpha-1}$ are taken, then the inequalities (2.1) become the inequalities via double conformable fractional integral*

$$\begin{aligned}
g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right)&\leq\tfrac{\alpha^2}{\left(a_2^\alpha-a_1^\alpha\right)\left(b_2^\alpha-b_1^\alpha\right)}\sum_{k=1}^{4}\hat{W}_k^1\\
&\leq\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}
\end{aligned}$$

where

$$\begin{aligned}
\hat{W}_1^1&=\int_{a_1}^{a_2}\int_{b_1}^{b_2}t^{\alpha-1}s^{\alpha-1}g\left(t,s\right)dsdt\\
&=\int_{a_1}^{a_2}\int_{b_1}^{b_2}g\left(t,s\right)d_\alpha s d_\alpha t,
\end{aligned}$$

$$\begin{aligned}\hat{W}_2^1 &= \int_{a_1}^{a_2}\int_{b_1}^{b_2} t^{\alpha-1}\left(b_2+b_1-s\right)^{\alpha-1} g\left(t,s\right) dsdt \\ &= \int_{a_1}^{a_2}\int_{b_1}^{b_2} g\left(t,b_2+b_1-v\right) d_\alpha v d_\alpha t,\end{aligned}$$

$$\begin{aligned}\hat{W}_3^1 &= \int_{a_1}^{a_2}\int_{b_1}^{b_2} \left(a_2+a_1-t\right)^{\alpha-1} s^{\alpha-1} g\left(t,s\right) dsdt \\ &= \int_{a_1}^{a_2}\int_{b_1}^{b_2} g\left(a_2+a_1-u,s\right) d_\alpha s d_\alpha u,\end{aligned}$$

and

$$\begin{aligned}\hat{W}_4^1 &= \int_{a_1}^{a_2}\int_{b_1}^{b_2} \left(a_2+a_1-t\right)^{\alpha-1}\left(b_2+b_1-s\right)^{\alpha-1} g\left(t,s\right) dsdt \\ &= \int_{a_1}^{a_2}\int_{b_1}^{b_2} g\left(a_2+a_1-u,b_2+b_1-v\right) d_\alpha v d_\alpha u.\end{aligned}$$

By using different method, two inequalities in Theorem 2.1 will be obtained as following forms:

Theorem 2.3. *Let $g{:}\triangle \subset \mathbb{R}^2 \to \mathbb{R}$ be co-ordinated convex on $\triangle := [a_1,a_2]\times[b_1,b_2]$ in with $a_1 < a_2$, $b_1 < b_2$ and $g \in L_1(\triangle)$. Then one has the inequalities:*

$$\begin{aligned}&g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right) \\ &\le \tfrac{1}{4\Lambda(1)}\left[{}_{a_1^+}I_\varphi g\left(a_2,\tfrac{b_1+b_2}{2}\right)+{}_{a_2^-}I_\varphi g\left(a_1,\tfrac{b_1+b_2}{2}\right)\right] \\ &+\tfrac{1}{4\Psi(1)}\left[+{}_{b_1^+}I_\psi g\left(\tfrac{a_1+a_2}{2},b_2\right)+{}_{b_2^-}I_\psi g\left(\tfrac{a_1+a_2}{2},b_1\right)\right] \\ &\le \tfrac{1}{4\Lambda(1)\Psi(1)}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi} g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}I_{\varphi,\psi} g\left(a_2,b_1\right)\right. \\ &\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi} g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}I_{\varphi,\psi} g\left(a_1,b_1\right)\right] \\ &\le \tfrac{1}{8\Lambda(1)}\left[{}_{a_1^+}I_\varphi g\left(a_2,b_1\right)+{}_{a_1^+}I_\varphi g\left(a_2,b_2\right)+{}_{a_2^-}I_\varphi g\left(a_1,b_1\right)+{}_{a_2^-}I_\varphi g\left(a_1,b_2\right)\right] \\ &+\tfrac{1}{8\Psi(1)}\left[{}_{b_1^+}I_\psi g\left(a_1,b_2\right)+{}_{b_1^+}I_\psi g\left(a_2,b_2\right)+{}_{b_2^-}I_\psi g\left(a_1,b_1\right)+{}_{b_2^-}I_\psi g\left(a_2,b_1\right)\right] \\ &\le \tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}.\end{aligned} \tag{2.4}$$

Proof. Since $g{:}\triangle \to \mathbb{R}$ is convex on the co-ordinates, it follows that the mapping $h_x{:}\ [b_1,b_2]\to\mathbb{R}$, $h_x(y) = g(x,y)$, is convex on $[b_1,b_2]$ for all $x \in [a_1,a_2]$, So the following Hermite Hadamard type inequality is obtained, for all $x \in [a_1,a_2]$,

$$h_x\left(\frac{b_1+b_2}{2}\right) \le \frac{1}{2\Psi(1)}\left[{}_{b_1^+}I_\varphi h_x\left(b_2\right)+{}_{b_2^-}I_\varphi h_x\left(b_1\right)\right] \le \frac{h_x\left(b_1\right)+h_x\left(b_2\right)}{2}.$$

That is,

$$\begin{aligned}&g\left(x,\tfrac{b_1+b_2}{2}\right) \\ &\le \tfrac{1}{2\Psi(1)}\left[\int_{b_1}^{b_2}\tfrac{\psi(b_2-y)}{b_2-y} g\left(x,y\right) dy+\int_{b_1}^{b_2}\tfrac{\psi(y-b_1)}{y-b_1} g\left(x,y\right) dy\right] \\ &\le \tfrac{g(x,b_1)+g(x,b_2)}{2}\end{aligned} \tag{2.5}$$

for all $x \in [a_1, a_2]$. Then multiplying both sides of (2.5) by $\frac{\varphi(a_2-x)}{a_2-x}$ and $\frac{\varphi(x-a_1)}{x-a_1}$, and integrating with respect to x over $[a_1, a_2]$, respectively, we have

$$\begin{aligned}
&\int_{a_1}^{a_2} \frac{\varphi(a_2-x)}{a_2-x} g\left(x, \frac{b_1+b_2}{2}\right) dx \\
&\leq \frac{1}{2\Psi(1)} \left[\int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\varphi(a_2-x)}{a_2-x}\frac{\psi(b_2-y)}{b_2-y} g(x,y)\, dydx \right. \\
&\left. + \int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\varphi(a_2-x)}{a_2-x}\frac{\psi(y-b_1)}{y-b_1} g(x,y)\, dydx\right] \\
&\leq \frac{1}{2}\left[\int_{a_1}^{a_2} \frac{\varphi(a_2-x)}{a_2-x} g(x,b_1)\, dx + \int_{a_1}^{a_2} \frac{\varphi(a_2-x)}{a_2-x} g(x,b_2)\, dx\right]
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\frac{\varphi(x-a_1)}{x-a_1} g\left(x, \frac{b_1+b_2}{2}\right) dx \\
&\leq \frac{1}{2\Psi(1)} \left[\int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\varphi(x-a_1)}{x-a_1}\frac{\psi(b_2-y)}{b_2-y} g(x,y)\, dydx \right. \\
&\left. + \int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\varphi(x-a_1)}{x-a_1}\frac{\psi(y-b_1)}{y-b_1} g(x,y)\, dydx\right] \\
&\leq \frac{1}{2}\left[\int_{a_1}^{a_2} \frac{\varphi(x-a_1)}{x-a_1} g(x,b_1)\, dx + \int_{a_1}^{a_2} \frac{\varphi(x-a_1)}{x-a_1} g(x,b_2)\, dx\right].
\end{aligned} \tag{2.7}$$

By similar argument applied for the mapping $h_y\colon [a_1, a_2] \to \mathbb{R}$, $h_y(x) = g(x,y)$, we have

$$\begin{aligned}
&\int_{b_1}^{b_2} \frac{\psi(b_2-y)}{b_2-y} g\left(\frac{a_1+a_2}{2}, y\right) dy \\
&\leq \frac{1}{2\Lambda(1)} \left[\int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\varphi(a_2-x)}{a_2-x}\frac{\psi(b_2-y)}{b_2-y} g(x,y)\, dydx \right. \\
&\left. + \int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\varphi(x-a_1)}{x-a_1}\frac{\psi(b_2-y)}{b_2-y} g(x,y)\, dydx\right] \\
&\leq \frac{1}{2}\left[\int_{b_1}^{b_2} \frac{\psi(b_2-y)}{b_2-y} g(a_1,y)\, dy + \int_{b_1}^{b_2} \frac{\psi(b_2-y)}{b_2-y} g(a_2,y)\, dy\right]
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\int_{b_1}^{b_2} \frac{\psi(y-b_1)}{y-b_1} g\left(\frac{a_1+a_2}{2}, y\right) dy \\
&\leq \frac{1}{2\Lambda(1)} \left[\int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\varphi(a_2-x)}{a_2-x}\frac{\psi(y-b_1)}{y-b_1} g(x,y)\, dydx \right. \\
&\left. + \int_{a_1}^{a_2}\int_{b_1}^{b_2} \frac{\psi(y-b_1)}{y-b_1}\frac{\psi(b_2-y)}{b_2-y} g(x,y)\, dydx\right] \\
&\leq \frac{1}{2}\left[\int_{b_1}^{b_2} \frac{\psi(y-b_1)}{y-b_1} g(a_1,y)\, dy + \int_{b_1}^{b_2} \frac{\psi(y-b_1)}{y-b_1} g(a_2,y)\, dy\right].
\end{aligned} \tag{2.9}$$

Adding the inequalities (2.6)-(2.9), we have

$$\begin{aligned}
&\tfrac{1}{\Lambda(1)}\left[{}_{a_1^+}I_{\varphi}g\left(a_2,\tfrac{b_1+b_2}{2}\right)+{}_{a_2^-}I_{\varphi}g\left(a_1,\tfrac{b_1+b_2}{2}\right)\right]\\
&+\tfrac{1}{\Psi(1)}\left[{}_{b_1^+}I_{\varphi}g\left(\tfrac{a_1+a_2}{2},b_2\right)+{}_{b_2^-}I_{\varphi}g\left(\tfrac{a_1+a_2}{2},b_1\right)\right]\\
&\times\int_0^1\tfrac{\varphi((a_2-a_1)t)}{t}dt\\
&\leq\tfrac{1}{\Lambda(1)\Psi(1)}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}I_{\varphi,\psi}g\left(a_2,b_1\right)\right.\\
&\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}I_{\varphi,\psi}g\left(a_1,b_1\right)\right]\\
&\leq\tfrac{1}{2\Lambda(1)}\left[{}_{a_1^+}I_{\varphi}g\left(a_2,b_1\right)+{}_{a_1^+}I_{\varphi}g\left(a_2,b_2\right)\right.\\
&\left.+{}_{a_2^-}I_{\varphi}g\left(a_1,b_1\right)+{}_{a_2^-}I_{\varphi}g\left(a_1,b_2\right)\right]\\
&+\tfrac{1}{2\Psi(1)}\left[{}_{b_1^+}I_{\varphi}g\left(a_1,b_2\right)+{}_{b_1^+}I_{\varphi}g\left(a_2,b_2\right)\right.\\
&\left.+{}_{b_2^-}I_{\varphi}g\left(a_1,b_1\right)+{}_{b_2^-}I_{\varphi}g\left(a_2,b_1\right)\right].
\end{aligned}$$

Thus, the second and third parts of the inequality (2.4) are obtained.

On the other hand, by using the first inequality in (1.10), we get

$$\begin{aligned}
&g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right)\\
&\leq\tfrac{1}{2\Lambda(1)}\left[\int_{a_1}^{a_2}\tfrac{\varphi(a_2-x)}{a_2-x}g\left(x,\tfrac{b_1+b_2}{2}\right)dx\right.\\
&\left.+\int_{a_1}^{a_2}\tfrac{\varphi(x-a_1)}{x-a_1}g\left(x,\tfrac{b_1+b_2}{2}\right)dx\right]
\end{aligned}$$

and

$$\begin{aligned}
&g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right)\\
&\leq\frac{1}{2\Psi(1)}\left[\int_{b_1}^{b_2}\tfrac{\psi(b_2-y)}{b_2-y}g\left(\tfrac{a_1+a_2}{2},y\right)dy\right.\\
&\left.+\int_{b_1}^{b_2}\tfrac{\psi(y-b_1)}{y-b_1}g\left(\tfrac{a_1+a_2}{2},y\right)dy\right].
\end{aligned}$$

Adding the above inequalities, we have

$$\begin{aligned}
&g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right)\\
&\leq\tfrac{1}{4\Lambda(1)}\left[{}_{a_1^+}I_{\varphi}g\left(a_2,\tfrac{b_1+b_2}{2}\right)+{}_{a_2^-}I_{\varphi}g\left(a_1,\tfrac{b_1+b_2}{2}\right)\right]\\
&+\tfrac{1}{4\Psi(1)}\left[{}_{b_1^+}I_{\varphi}g\left(\tfrac{a_1+a_2}{2},b_2\right)+{}_{b_2^-}I_{\varphi}g\left(\tfrac{a_1+a_2}{2},b_1\right)\right]
\end{aligned}$$

which gives the first part of inequality (2.4).

Finally, from the second part of inequality (1.10), we get the following;

$$\begin{aligned}
&\tfrac{1}{2\Lambda(1)}\left[\int_{a_1}^{a_2}\tfrac{\varphi(a_2-x)}{a_2-x}g\left(x,b_1\right)dx+\int_{a_1}^{a_2}\tfrac{\varphi(x-a_1)}{x-a_1}g\left(x,b_1\right)dx\right]\\
&\leq\tfrac{g(a_1,b_1)+g(a_2,b_1)}{2},\\
&\tfrac{1}{2\Lambda(1)}\left[\int_{a_1}^{a_2}\tfrac{\varphi(a_2-x)}{a_2-x}g\left(x,b_2\right)dx+\int_{a_1}^{a_2}\tfrac{\varphi(x-a_1)}{x-a_1}g\left(x,b_2\right)dx\right]\\
&\leq\tfrac{g(a_1,b_2)+g(a_2,b_2)}{2},
\end{aligned}$$

$$\frac{1}{2\Psi(1)}\left[\int_{b_1}^{b_2}\frac{\psi(b_2-y)}{b_2-y}g(a_1,y)\,dy+\int_{b_1}^{b_2}\frac{\psi(y-b_1)}{y-b_1}g(a_1,y)\,dy\right]$$
$$\leq\frac{g(a_1,b_1)+g(a_1,b_2)}{2}$$

and

$$\frac{1}{2\Psi(1)}\left[\int_{b_1}^{b_2}\frac{\psi(b_2-y)}{b_2-y}g(a_2,y)\,dy+\int_{b_1}^{b_2}\frac{\psi(y-b_1)}{y-b_1}g(a_2,y)\,dy\right]$$
$$\leq\frac{g(a_2,b_1)+g(a_2,b_2)}{2},$$

which give, by addition, the last inequality in (2.4),

$$\frac{1}{8\Lambda(1)}\left[{}_{a_1^+}I_{\varphi}g(a_2,b_1)+{}_{a_1^+}I_{\varphi}g(a_2,b_2)+{}_{a_2^-}I_{\varphi}g(a_1,b_1)+{}_{a_2^-}I_{\varphi}g(a_1,b_2)\right]$$
$$+\frac{1}{8\Psi(1)}\left[{}_{b_1^+}I_{\psi}g(a_1,b_2)+{}_{b_1^+}I_{\psi}g(a_2,b_2)+{}_{b_2^-}I_{\psi}g(a_1,b_1)+{}_{b_2^-}I_{\psi}g(a_2,b_1)\right]$$
$$\leq\frac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}.$$

□

Remark. *Under the hypotheses of Theorem 2.3, when $\varphi(t)=t$ and $\psi(s)=s$ are taken, (2.4) and (1.2) become the same expression.*

Remark. *Under the hypotheses of Theorem 2.3, when $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^{\beta}}{\Gamma(\beta)}$ are taken, Theorem 2.3 and Theorem 4 in* [20] *become the same expression;*

$$\begin{aligned}
g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right) &\leq\frac{\Gamma(\alpha+1)}{4(a_2-a_1)^{\alpha}}\left[J_{a_1^+}^{\alpha}g\left(a_2,\tfrac{b_1+b_2}{2}\right)+J_{a_2^-}^{\alpha}g\left(a_1,\tfrac{b_1+b_2}{2}\right)\right]\\
&+\frac{\Gamma(\beta+1)}{4(b_2-b_1)^{\beta}}\left[J_{b_1^+}^{\beta}g\left(\tfrac{a_1+a_2}{2},b_2\right)+J_{b_2^-}^{\beta}g\left(\tfrac{a_1+a_2}{2},b_1\right)\right]\\
&\leq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(a_2-a_1)^{\alpha}(b_2-b_1)^{\beta}}\left[J_{a_1^+,b_1^+}^{\alpha,\beta}g(a_2,b_2)+J_{a_1^+,b_2^-}^{\alpha,\beta}g(a_2,b_1)\right.\\
&\left.+J_{a_2^-,b_1^+}^{\alpha,\beta}g(a_1,b_2)+J_{a_2^-,b_2^-}^{\alpha,\beta}g(a_1,b_1)\right]\\
&\leq\frac{\Gamma(\alpha+1)}{8(a_2-a_1)^{\alpha}}\left[J_{a_1^+}^{\alpha}g(a_2,b_1)+J_{a_1^+}^{\alpha}g(a_2,b_2)\right.\\
&\left.+J_{a_2^-}^{\alpha}g(a_1,b_1)+J_{a_2^-}^{\alpha}g(a_1,b_2)\right]\\
&+\frac{\Gamma(\beta+1)}{8(b_2-b_1)^{\beta}}\left[J_{b_1^+}^{\beta}g(a_1,b_2)+J_{b_1^+}^{\beta}g(a_2,b_2)\right.\\
&\left.+J_{b_2^-}^{\beta}g(a_1,b_1)+J_{b_2^-}^{\beta}g(a_2,b_1)\right]\\
&\leq\frac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}.
\end{aligned}$$

Remark. *Under the hypotheses of Theorem 2.3, when $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ are taken, Theorem 2.3 and Theorem 3.7 in* [7] *become the same expression.*

Corollary 2.4. *Under the hypotheses of Theorem 2.3, when $\varphi(t)=t(a_2-t)^{\alpha-1}$, $\psi(s)=s(b_2-s)^{\alpha-1}$, the inequalities (2.4) become the inequalities via double conformable fractional integral;*

$$
\begin{aligned}
&g\left(\tfrac{a_1+a_2}{2},\tfrac{b_1+b_2}{2}\right)\\
&\leq \tfrac{\alpha}{4\left(a_2^\alpha-a_1^\alpha\right)}\left[\int_{a_1}^{a_2} g\left(t,\tfrac{b_1+b_2}{2}\right)d_\alpha t+\int_{a_1}^{a_2} g\left(a_2+a_1-u,\tfrac{b_1+b_2}{2}\right)d_\alpha u\right]\\
&+\tfrac{\alpha}{4\left(b_2^\alpha-b_1^\alpha\right)}\left[\int_{a_1}^{a_2} g\left(\tfrac{a_1+a_2}{2},s\right)d_\alpha s-\int_{a_1}^{a_2} g\left(\tfrac{a_1+a_2}{2},b_2+b_1-v\right)d_\alpha v\right]\\
&\leq \tfrac{\alpha^2}{\left(a_2^\alpha-a_1^\alpha\right)\left(b_2^\alpha-b_1^\alpha\right)}\left[\int_{a_1}^{a_2}\int_{b_1}^{b_2} g\left(t,s\right)d_\alpha s d_\alpha t+\int_{a_1}^{a_2}\int_{b_1}^{b_2} g\left(t,b_2+b_1-v\right)d_\alpha v d_\alpha t\right.\\
&\left.+\int_{a_1}^{a_2}\int_{b_1}^{b_2} g\left(a_2+a_1-u,s\right)d_\alpha s d_\alpha u+\int_{a_1}^{a_2}\int_{b_1}^{b_2} g\left(a_2+a_1-u,b_2+b_1-v\right)d_\alpha v d_\alpha u\right]\\
&\leq \tfrac{\alpha}{8\left(a_2^\alpha-a_1^\alpha\right)}\left[\int_{a_1}^{a_2}\left[g\left(t,b_1\right)+g\left(t,b_2\right)\right]d_\alpha t\right.\\
&\left.+\int_{a_1}^{a_2}\left[g\left(a_2+a_1-u,b_1\right)+g\left(a_2+a_1-u,b_2\right)\right]d_\alpha u\right]\\
&+\tfrac{\alpha}{4\left(b_2^\alpha-b_1^\alpha\right)}\left[\int_{b_1}^{b_2}\left[g\left(a_1,s\right)+g\left(a_2,s\right)\right]d_\alpha s\right.\\
&\left.+\int_{b_1}^{b_2}\left[g\left(a_1,b_2+b_1-v\right)+g\left(a_2,b_2+b_1-v\right)\right]d_\alpha v\right]\\
&\leq \tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}.
\end{aligned}
$$

3. Applications For Some New Trapezoid Inequalieties

The purpose of this section is to obtain new inequalities including some Hermite-Hadamard type inequalities with the help of our new double fractional integral. First, we give our first conclusion for this part. For this, we need the following auxiliary Lemma.

Lemma 3.1. *Let $g:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\triangle:=[a_1,a_2]\times[b_1,b_2]$ in $\mathbb{R}^2$ with $a_1<a_2$, $b_1<b_2$. If $\frac{\partial^2 g}{\partial t\partial s}\in L_1(\triangle)$, then the following equality holds:*

$$
\begin{aligned}
&\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\\
&+\tfrac{1}{4\Lambda(1)\Psi(1)}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}I_{\varphi,\psi}g\left(a_2,b_1\right)\right.\\
&\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}I_{\varphi,\psi}g\left(a_1,b_1\right)\right]-A\\
&=\tfrac{(a_2-a_1)(b_2-b_1)}{4\Lambda(1)\Psi(1)}\\
&\times\left\{\int_0^1\int_0^1\Lambda(t)\Psi(s)\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)a_2,sb_1+(1-s)b_2\right)dsdt\right.\\
&-\int_0^1\int_0^1\Lambda(1-t)\Psi(s)\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)a_2,sb_1+(1-s)b_2\right)dsdt\\
&-\int_0^1\int_0^1\Lambda(t)\Psi(1-s)\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)a_2,\ sb_1+(1-s)b_2\right)dsdt\\
&\left.+\int_0^1\int_0^1\Lambda(1-t)\Psi(1-s)\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)a_2,\ sb_1+(1-s)b_2\right)dsdt\right\}
\end{aligned}
\tag{3.1}
$$

where

$$\begin{aligned} A &= \tfrac{1}{4\Psi(1)}\left[{}_{b_1^+}I_{\varphi}g\left(a_1,b_2\right)+{}_{b_1^+}I_{\varphi}g\left(a_2,b_2\right)\right. \\ &\left.+{}_{b_2^-}I_{\varphi}g\left(a_1,b_1\right)+{}_{b_2^-}I_{\varphi}g\left(a_2,b_1\right)\right] \\ &+\tfrac{1}{4\Lambda(1)}[\left[{}_{a_1^+}I_{\varphi}g\left(a_2,b_1\right)+{}_{a_1^+}I_{\varphi}g\left(a_2,b_2\right)\right. \\ &\left.+{}_{a_2^-}I_{\varphi}g\left(a_1,b_1\right)+{}_{a_2^-}I_{\varphi}g\left(a_1,b_2\right)\right]. \end{aligned} \tag{3.2}$$

Proof. Under Lemma's hypotheses, if we calculate the following integrals;

$$\begin{aligned} I_1 &= \int_0^1\int_0^1 \Lambda(t)\Psi(s)\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)a_2,sb_1+(1-s)b_2\right)dsdt \\ &= \tfrac{1}{(a_2-a_1)(b_2-b_1)}\Lambda(1)\Psi(1)g\left(a_1,b_1\right) \\ &-\tfrac{1}{(a_2-a_1)(b_2-b_1)}\Psi(1) \\ &\times\int_0^1 \tfrac{\varphi((a_2-a_1)t)}{t}g\left(ta_1+(1-t)a_2,b_1\right)dt \\ &-\tfrac{1}{(a_2-a_1)(b_2-b_1)}\Lambda(1) \\ &\times\int_0^1 \tfrac{\psi((b_2-b_1)s)}{s}g\left(a_1,sb_1+(1-s)b_2\right)ds \\ &+\tfrac{1}{(a_2-a_1)(b_2-b_1)} \\ &\times\int_0^1\int_0^1 \tfrac{\varphi((a_2-a_1)t)}{t}\tfrac{\psi((b_2-b_1)s)}{s} \\ &\times g\left(ta_1+(1-t)a_2,sb_1+(1-s)b_2\right)dsdt \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} I_2 &= \int_0^1\int_0^1 \Lambda(1-t)\Psi(s)\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)a_2,sb_1+(1-s)b_2\right)dsdt \\ &= -\tfrac{1}{(a_2-a_1)(b_2-b_1)}\Psi(1)\Lambda(1)g\left(a_2,b_1\right) \\ &+\tfrac{1}{(a_2-a_1)(b_2-b_1)}\Psi(1) \\ &\times\int_0^1 \tfrac{\varphi((a_2-a_1)(1-t))}{(1-t)}g\left(ta_1+(1-t)a_2,b_1\right)dt \\ &+\tfrac{1}{(a_2-a_1)(b_2-b_1)}\Lambda(1) \\ &\times\int_0^1 \tfrac{\psi((b_2-b_1)s)}{s}g\left(a_2,sb_1+(1-s)b_2\right)ds \\ &-\tfrac{1}{(a_2-a_1)(b_2-b_1)} \\ &\times\int_0^1\int_0^1 \tfrac{\varphi((a_2-a_1)(1-t))}{(1-t)}\tfrac{\psi((b_2-b_1)s)}{s} \\ &\times g\left(ta_1+(1-t)a_2,sb_1+(1-s)b_2\right)dsdt \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} I_3 &= \int_0^1\int_0^1 \Lambda(t)\Psi(1-s)\frac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)\,a_2, sb_1+(1-s)\,b_2\right)dsdt \\ &= -\frac{1}{(a_2-a_1)(b_2-b_1)}\Lambda(1)\Psi(1)g\left(a_1,b_2\right) \\ &+\frac{1}{(a_2-a_1)(b_2-b_1)}\Psi(1) \\ &\times\int_0^1 \frac{\varphi((a_2-a_1)t)}{t}g\left(ta_1+(1-t)\,a_2,b_2\right)dt \\ &+\frac{1}{(a_2-a_1)(b_2-b_1)}\Lambda(1) \\ &\times\int_0^1 \frac{\varphi((b_2-b_1)(1-s))}{(1-s)}g\left(a_1,sb_1+(1-s)\,b_2\right)ds \\ &-\frac{1}{(a_2-a_1)\,(b_2-b_1)} \\ &\times\int_0^1\int_0^1 \frac{\varphi((a_2-a_1)t)}{t}\frac{\varphi((b_2-b_1)(1-s))}{(1-s)} \\ &\times g\left(ta_1+(1-t)\,a_2, sb_1+(1-s)\,b_2\right)dsdt \end{aligned} \tag{3.5}$$

and finally

$$\begin{aligned} I_4 &= \int_0^1\int_0^1 \Lambda(1-t)\Psi(1-s)\frac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)\,a_2, sb_1+(1-s)\,b_2\right)dsdt \\ &= \frac{1}{(a_2-a_1)(b_2-b_1)}\Lambda(1)\Psi(1)g\left(a_2,b_2\right) \\ &-\frac{1}{(a_2-a_1)(b_2-b_1)}\Psi(1) \\ &\times\int_0^1 \frac{\varphi((a_2-a_1)(1-t))}{(1-t)}g\left(ta_1+(1-t)\,a_2,b_2\right)dt \\ &-\frac{1}{(a_2-a_1)(b_2-b_1)}\Lambda(1) \\ &\times\int_0^1 \frac{\varphi((b_2-b_1)(1-s))}{(1-s)}g\left(a_2,sb_1+(1-s)\,b_2\right)ds \\ &+\frac{1}{(a_2-a_1)(b_2-b_1)} \\ &\times\int_0^1\int_0^1 \frac{\varphi((a_2-a_1)(1-t))}{(1-t)}\frac{\varphi((b_2-b_1)(1-s))}{(1-s)} \\ &\times g\left(ta_1+(1-t)\,a_2, sb_1+(1-s)\,b_2\right)dsdt \end{aligned} \tag{3.6}$$

we obtain integrals I_1, I_2, I_3, I_4 respectively.From (3.3)-(3.6), using the change of the variables as $x = ta_1+(1-t)\,a_2$ and $y = sb_1+(1-s)\,b_2$ for $t,s\in[0,1]^2$, we can write

$$\begin{aligned} &I_1 - I_2 - I_3 + I_4 \\ &= \frac{[g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)]}{(a_2-a_1)(b_2-b_1)}\Lambda(1)\Psi(1) \\ &-\frac{1}{(a_2-a_1)(b_2-b_1)}\Lambda(1)\left[{}_{b_1^+}I_{\varphi}g\left(a_1,b_2\right)+{}_{b_1^+}\,I_{\varphi}g\left(a_2,b_2\right)\right. \\ &\left.+{}_{b_2^-}I_{\varphi}g\left(a_1,b_1\right)+{}_{b_2^-}\,I_{\varphi}g\left(a_2,b_1\right)\right] \\ &-\frac{1}{(a_2-a_1)(b_2-b_1)}\Psi(1)\left[{}_{a_1^+}I_{\varphi}g\left(a_2,b_1\right)+{}_{a_1^+}\,I_{\varphi}g\left(a_2,b_2\right)\right. \\ &\left.+{}_{a_2^-}I_{\varphi}g\left(a_1,b_1\right)+{}_{a_2^-}\,I_{\varphi}g\left(a_1,b_2\right)\right] \\ &+\frac{1}{(a_2-a_1)(b_2-b_1)} \\ &\times\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}\,I_{\varphi,\psi}g\left(a_2,b_1\right)\right. \\ &\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}\,I_{\varphi,\psi}g\left(a_1,b_1\right)\right]. \end{aligned} \tag{3.7}$$

Multiplying the both sides of (3.7) by $(a_2-a_1)(b_2-b_1)/4\Lambda(1)\Psi(1)$, we obtain (3.1), which completes the proof. □

Remark. *Under the hypotheses of Lemma 3.1, when* $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ *and* $\psi(s)=\frac{s^{\beta}}{\Gamma(\beta)}$ *are taken, Lemma 3.1 and Lemma 2 in* [20] *become the same expression;*

$$\begin{aligned}
&\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(a_2-a_1)^{\alpha}(b_2-b_1)^{\beta}}\left[J^{\alpha,\beta}_{a_1^+,b_1^+}g(a_2,b_2)+J^{\alpha,\beta}_{a_1^+,b_2^-}g(a_2,b_1)\right.\\
&\left.+J^{\alpha,\beta}_{a_2^-,b_1^+}g(a_1,b_2)+J^{\alpha,\beta}_{a_2^-,b_2^-}g(a_1,b_1)\right]-A_1\\
&=\tfrac{(a_2-a_1)(b_2-b_1)}{4}\\
&\times\left\{\int_0^1\int_0^1 t^{\frac{\alpha}{k}}s^{\frac{\beta}{k}}\tfrac{\partial^2 g}{\partial t\partial s}(ta_1+(1-t)a_2,sb_1+(1-s)b_2)\,dsdt\right.\\
&-\int_0^1\int_0^1(1-t)\tfrac{\alpha}{k}s^{\frac{\beta}{k}}\tfrac{\partial^2 g}{\partial t\partial s}(ta_1+(1-t)a_2,sb_1+(1-s)b_2)\,dsdt\\
&-\int_0^1\int_0^1 t\tfrac{\alpha}{k}(1-s)^{\frac{\beta}{k}}\tfrac{\partial^2 g}{\partial t\partial s}(ta_1+(1-t)a_2,\ sb_1+(1-s)b_2)\,dsdt\\
&\left.+\int_0^1\int_0^1(1-t)\tfrac{\alpha}{k}(1-s)^{\frac{\beta}{k}}\tfrac{\partial^2 g}{\partial t\partial s}(ta_1+(1-t)a_2,\ sb_1+(1-s)b_2)\,dsdt\right\}
\end{aligned}$$

where

$$\begin{aligned}
&A_1\\
&=\tfrac{\Gamma(\beta+1)}{4(b_2-b_1)^{\beta}}\left[J^{\beta}_{b_1^+}g(a_1,b_2)+J^{\beta}_{b_1^+}g(a_2,b_2)+J^{\beta}_{b_2^-}g(a_1,b_1)+J^{\beta}_{b_2^-}g(a_2,b_1)\right]\\
&+\tfrac{\Gamma(\alpha+1)}{4(a_2-a_1)^{\alpha}}[J^{\alpha}_{a_1^+}g(a_2,b_1)+J^{\alpha}_{a_1^+}g(a_2,b_2)+J^{\alpha}_{a_2^-}g(a_1,b_1)+J^{\alpha}_{a_2^-}g(a_1,b_2).
\end{aligned}\tag{3.8}$$

Under the hypotheses of Lemma 3.1, if we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$then the above equality (3.1) reduce to the following inequality involving k-fractional integrals

$$\begin{aligned}
&\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\\
&+\tfrac{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{4(a_2-a_1)^{\frac{\alpha}{k}}(b_2-b_1)^{\frac{\beta}{k}}}\left[{}_kI^{\alpha,\beta}_{a_1^+,b_1^+}g(a_2,b_2)+{}_k\,I^{\alpha,\beta}_{a_1^+,b_2^-}g(a_2,b_1)\right.\\
&\left.+{}_kI^{\alpha,\beta}_{a_2^-,b_1^+}g(a_1,b_2)+{}_k\,I^{\alpha,\beta}_{a_2^-,b_2^-}g(a_1,b_1)\right]-A_2\\
&=\tfrac{(a_2-a_1)(b_2-b_1)}{4}\\
&\times\left\{\int_0^1\int_0^1 t^{\alpha}s^{\beta}\tfrac{\partial^2 g}{\partial t\partial s}(ta_1+(1-t)a_2,sb_1+(1-s)b_2)\,dsdt\right.\\
&-\int_0^1\int_0^1(1-t)^{\alpha}s^{\beta}\tfrac{\partial^2 g}{\partial t\partial s}(ta_1+(1-t)a_2,sb_1+(1-s)b_2)\,dsdt\\
&-\int_0^1\int_0^1 t^{\alpha}(1-s)^{\beta}\tfrac{\partial^2 g}{\partial t\partial s}(ta_1+(1-t)a_2,\ sb_1+(1-s)b_2)\,dsdt\\
&\left.+\int_0^1\int_0^1(1-t)^{\alpha}(1-s)^{\beta}\tfrac{\partial^2 g}{\partial t\partial s}(ta_1+(1-t)a_2,\ sb_1+(1-s)b_2)\,dsdt\right\}
\end{aligned}$$

where

$$\begin{aligned}
&A \\
&= \tfrac{\Gamma_k(\beta+k)}{4(b_2-b_1)^{\frac{\beta}{k}}}\left[{}_kI^{\beta}_{b_1^+}g\left(a_1,b_2\right)+{}_k\,I^{\beta}_{b_1^+}g\left(a_2,b_2\right)\right. \\
&\left.+{}_kI^{\beta}_{b_2^-}g\left(a_1,b_1\right)+{}_k\,I^{\beta}_{b_2^-}g\left(a_2,b_1\right)\right] \\
&+\tfrac{\Gamma_k(\alpha+k)}{4(a_2-a_1)^{\frac{\alpha}{k}}}\left[{}_kI^{\alpha}_{a_1^+}g\left(a_2,b_1\right)+{}_k\,I^{\alpha}_{a_1^+}g\left(a_2,b_2\right)\right. \\
&\left.+{}_kI^{\alpha}_{a_2^-}g\left(a_1,b_1\right)+{}_k\,I^{\alpha}_{a_2^-}g\left(a_1,b_2\right)\right].
\end{aligned} \tag{3.9}$$

Next, we start to state the first theorem containing the Hermite-Hadamard-type inequality for double fractional integrals.

Theorem 3.2. *Let $g:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\triangle:=[a_1,a_2]\times[b_1,b_2]$ in $\mathbb{R}^2$ with $a_1<a_2$, $b_1<b_2$. If $\left|\frac{\partial^2 g}{\partial t\partial s}\right|$ is a convex function on the co-ordinates on $\triangle$, then one has the inequalities:*

$$\begin{aligned}
&\left|\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\right. \\
&+\tfrac{1}{4\Lambda(1)\Psi(1)}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}\,I_{\varphi,\psi}g\left(a_2,b_1\right)\right. \\
&\left.\left.+\,{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}\,I_{\varphi,\psi}g\left(a_1,b_1\right)\right]-A\right| \\
&\leq \tfrac{(a_2-a_1)(b_2-b_1)}{4\Lambda(1)\Psi(1)}\int_0^1\int_0^1\Lambda(t)\Psi(s)dsdt \\
&\times\left\{\left|\frac{\partial^2 g}{\partial t\partial s}\left(a_1,b_1\right)\right|+\left|\frac{\partial^2 g}{\partial t\partial s}\left(a_2,b_1\right)\right|+\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_1,b_2\right)\right|+\left|\frac{\partial^2 g}{\partial t\partial s}\left(a_2,b_2\right)\right|\right\}
\end{aligned} \tag{3.10}$$

where A is defined by (3.2).

Proof. From Lemma 3.1, we have the following inequality

$$\begin{aligned}
&\left|\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\right. \\
&+\tfrac{1}{4\Lambda(1)\Psi(1)}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}\,I_{\varphi,\psi}g\left(a_2,b_1\right)\right. \\
&\left.\left.+\,{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}\,I_{\varphi,\psi}g\left(a_1,b_1\right)\right]-A\right| \\
&\leq \tfrac{(a_2-a_1)(b_2-b_1)}{4\Lambda(1)\Psi(1)} \\
&\times\left\{\int_0^1\int_0^1\Lambda(t)\Psi(s)\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)\,a_2,sb_1+(1-s)\,b_2\right)\right|dsdt\right. \\
&+\int_0^1\int_0^1\Lambda(1-t)\Psi(s)\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)\,a_2,sb_1+(1-s)\,b_2\right)\right|dsdt \\
&+\int_0^1\int_0^1\Lambda(t)\Psi(1-s)\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)\,a_2,sb_1+(1-s)\,b_2\right)\right|dsdt \\
&+\int_0^1\int_0^1\Lambda(1-t)\Psi(1-s) \\
&\left.\times\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)\,a_2,sb_1+(1-s)\,b_2\right)\right|dsdt\right\}.
\end{aligned} \tag{3.11}$$

Since $\left|\frac{\partial^2 g}{\partial t \partial s}\right|$ is a convex function on the co-ordinates on $\triangle$, then one has:

$$\begin{aligned}
&\left|\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\right. \\
&+\tfrac{1}{4\Lambda(1)\Psi(1)}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}I_{\varphi,\psi}g\left(a_2,b_1\right)\right. \\
&\left.\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}I_{\varphi,\psi}g\left(a_1,b_1\right)\right]-A\right| \\
&\leq \tfrac{(a_2-a_1)(b_2-b_1)}{4\Lambda(1)\Psi(1)} \\
&\times\int_0^1\int_0^1\left[\Lambda(t)\Psi(s)+\Lambda(1-t)\Psi(s)+\Lambda(t)\Psi(1-s)+\Lambda(1-t)\Psi(1-s)\right] \\
&\times\left\{ts\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_1,b_1\right)\right|+s\left(1-t\right)\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_2,b_1\right)\right|dt\right. \\
&\left.+t\left(1-s\right)\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_1,b_2\right)\right|+\left(1-s\right)\left(1-t\right)\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_2,b_2\right)\right|ds\right\} \\
&=\tfrac{(a_2-a_1)(b_2-b_1)}{4\Lambda(1)\Psi(1)}\int_0^1\int_0^1\Lambda(t)\Psi(s)dsdt \\
&\times\left\{\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_1,b_1\right)\right|+\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_2,b_1\right)\right|+\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_1,b_2\right)\right|+\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_2,b_2\right)\right|\right\}.
\end{aligned}$$

This completes the proof of the inequality (3.10). □

Remark. *Under the hypotheses of Theorem 3.2, when* $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ *and* $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ *are taken, Theorem 3.2 and Theorem5 in* [20] *become the same expression;*

$$\begin{aligned}
&\left|\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\right. \\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(a_2-a_1)^\alpha(b_2-b_1)^\beta} \\
&\times\left[J^{\alpha,\beta}_{a_1^+,b_1^+}g\left(a_2,b_2\right)+J^{\alpha,\beta}_{a_1^+,b_2^-}g\left(a_2,b_1\right)\right. \\
&\left.\left.+J^{\alpha,\beta}_{a_2^-,b_1^+}g\left(a_1,b_2\right)+J^{\alpha,\beta}_{a_2^-,b_2^-}g\left(a_1,b_1\right)\right]-A_1\right| \\
&\leq\tfrac{(a_2-a_1)(b_2-b_1)}{4\left(\frac{\alpha}{k}+1\right)\left(\frac{\beta}{k}+1\right)} \\
&\times\left(\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_1,b_1\right)\right|+\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_1,b_2\right)\right|+\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_2,b_1\right)\right|+\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_2,b_2\right)\right|\right)
\end{aligned}$$

where A_1 *is defined by (3.8).*

Remark. *Under the hypotheses of Theorem 2.1, if we take* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ *then we have the following Hermite-Hadamard type inequality for k-fractional integrals;*

$$\begin{aligned}
&\left|\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\right. \\
&+\tfrac{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{4(a_2-a_1)^{\frac{\alpha}{k}}(b_2-b_1)^{\frac{\beta}{k}}}\left[{}_kI^{\alpha,\beta}_{a_1^+,b_1^+}g\left(a_2,b_2\right)+{}_k\,I^{\alpha,\beta}_{a_1^+,b_2^-}g\left(a_2,b_1\right)\right. \\
&\left.\left.+{}_kI^{\alpha,\beta}_{a_2^-,b_1^+}g\left(a_1,b_2\right)+{}_k\,I^{\alpha,\beta}_{a_2^-,b_2^-}g\left(a_1,b_1\right)\right]-A_2\right| \\
&\leq\tfrac{(a_2-a_1)(b_2-b_1)}{4(\alpha+1)(\beta+1)}\left(\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_1,b_1\right)\right|+\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_1,b_2\right)\right|\right. \\
&\left.+\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_2,b_1\right)\right|+\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_2,b_2\right)\right|\right)
\end{aligned}$$

where A_2 is defined by (3.9).

Theorem 3.3. *Let $g{:}\triangle \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\triangle := [a_1, a_2] \times [b_1, b_2]$ in $\mathbb{R}^2$ with $a_1 < a_2$, $b_1 < b_2$. If $\left|\frac{\partial^2 g}{\partial t \partial s}\right|^q$, $q > 1$, is a convex function on the co-ordinates on $\triangle$, then one has the inequalities:*

$$\begin{aligned}
&\left|\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\right.\\
&+\tfrac{1}{4\Lambda(1)\Psi(1)}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}I_{\varphi,\psi}g\left(a_2,b_1\right)\right.\\
&\left.\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}I_{\varphi,\psi}g\left(a_1,b_1\right)\right]-A\right|\\
&\leq \tfrac{(a_2-a_1)(b_2-b_1)}{\Lambda(1)\Psi(1)}\left(\tfrac{1}{4}\right)^{1/q}\left(\int_0^1\int_0^1\left(\Lambda(t)\right)^p\left(\Psi(s)\right)^p dsdt\right)^{\frac{1}{p}}\\
&\times\left(\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_1,b_1\right)\right|^q+\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_1,b_2\right)\right|^q+\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_2,b_1\right)\right|^q+\left|\tfrac{\partial^2 g}{\partial s\partial t}\left(a_2,b_2\right)\right|^q\right)^{1/q}
\end{aligned}$$

where A is defined by (3.2).

Proof. From Lemma 3.1, we have inequality (3.11). By using the well known Hölder's inequality for double integrals, we get

$$\begin{aligned}
&\left|\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\right.\\
&+\tfrac{1}{4\Lambda(1)\Psi(1)}\left[{}_{a_1^+,b_1^+}I_{\varphi,\psi}g\left(a_2,b_2\right)+{}_{a_1^+,b_2^-}I_{\varphi,\psi}g\left(a_2,b_1\right)\right.\\
&\left.\left.+{}_{a_2^-,b_1^+}I_{\varphi,\psi}g\left(a_1,b_2\right)+{}_{a_2^-,b_2^-}I_{\varphi,\psi}g\left(a_1,b_1\right)\right]-A\right|\\
&\leq \tfrac{(a_2-a_1)(b_2-b_1)}{4\Lambda(1)\Psi(1)}\left\{\left[\int_0^1\int_0^1\left(\Lambda(t)\right)^p\left(\Psi(s)\right)^p dsdt\right]^{\frac{1}{p}}\right.\\
&+\left[\int_0^1\int_0^1\left(\Lambda(1-t)\right)^p\left(\Psi(s)\right)^p dsdt\right]^{\frac{1}{p}}\\
&+\left[\int_0^1\int_0^1\left(\Lambda(t)\right)^p\left(\Psi(1-s)\right)^p dsdt\right]^{\frac{1}{p}}\\
&+\left[\int_0^1\int_0^1\left(\Lambda(1-t)\right)^p\left(\Psi(1-s)\right)^p dsdt\right]^{\frac{1}{p}}\\
&\left.+\left[\int_0^1\int_0^1\left(\Lambda(1-t)\right)^p\left(\Psi(1-s)\right)^p dsdt\right]^{\frac{1}{p}}\right\}\\
&\times\left(\int_0^1\int_0^1\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)a_2,sb_1+(1-s)b_2\right)\right|^q dtds\right)^{1/q}.
\end{aligned}$$

Since $\left|\frac{\partial^2 g}{\partial t\partial s}\right|^q$ is a convex function on the co-ordinates on $\triangle$, we know that for $t \in [0,1]$

$$\begin{aligned}
&\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(ta_1+(1-t)a_2,sb_1+(1-s)b_2\right)\right|^q\\
&\leq ts\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_1,b_1\right)\right|^q+t\left(1-s\right)\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_1,b_2\right)\right|^q\\
&+\left(1-t\right)s\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_2,b_1\right)\right|^q+\left(1-t\right)\left(1-s\right)\left|\tfrac{\partial^2 g}{\partial t\partial s}\left(a_2,b_2\right)\right|^q.
\end{aligned}$$

By the simple calculating we obtain the required result. □

Remark. *Under the hypotheses of Theorem 3.3, when* $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ *and* $\psi(s)=\frac{s^{\beta}}{\Gamma(\beta)}$ *are taken, Theorem 3.3 and Theorem 6 in* [20] *become the same expression;*

$$\begin{aligned}
&\left|\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\right.\\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(a_2-a_1)^{\alpha}(b_2-b_1)^{\beta}}\left[J^{\alpha,\beta}_{a_1^+,b_1^+}g(a_2,b_2)+J^{\alpha,\beta}_{a_1^+,b_2^-}g(a_2,b_1)\right.\\
&\left.\left.+J^{\alpha,\beta}_{a_2^-,b_1^+}g(a_1,b_2)+J^{\alpha,\beta}_{a_2^-,b_2^-}g(a_1,b_1)\right]-A\right|\\
&\leq\tfrac{(a_2-a_1)(b_2-b_1)}{[(\alpha p+1)(\beta p+1)]^{1/p}}\left(\tfrac{1}{4}\right)^{1/q}\\
&\times\left(\left|\tfrac{\partial^2 g}{\partial s\partial t}(a_1,b_1)\right|^q+\left|\tfrac{\partial^2 g}{\partial s\partial t}(a_1,b_2)\right|^q+\left|\tfrac{\partial^2 g}{\partial s\partial t}(a_2,b_1)\right|^q+\left|\tfrac{\partial^2 g}{\partial s\partial t}(a_2,b_2)\right|^q\right)^{1/q}
\end{aligned}$$

where A *is defined by (3.2).*

Remark. *Under the hypotheses of Theorem 3.3, if we take* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ *then we have the following Hermite-Hadamard type inequality for* k*-fractional integrals*

$$\begin{aligned}
&\left|\tfrac{g(a_1,b_1)+g(a_1,b_2)+g(a_2,b_1)+g(a_2,b_2)}{4}\right.\\
&+\tfrac{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{4(a_2-a_1)^{\frac{\alpha}{k}}(b_2-b_1)^{\frac{\beta}{k}}}\left[{}_kI^{\alpha,\beta}_{a_1^+,b_1^+}g(a_2,b_2)+{}_k\,I^{\alpha,\beta}_{a_1^+,b_2^-}g(a_2,b_1)\right.\\
&\left.\left.+{}_kI^{\alpha,\beta}_{a_2^-,b_1^+}g(a_1,b_2)+{}_k\,I^{\alpha,\beta}_{a_2^-,b_2^-}g(a_1,b_1)\right]-A_2\right|\\
&\leq\tfrac{(a_2-a_1)(b_2-b_1)}{\left[\left(\frac{\alpha p}{k}+1\right)\left(\frac{\beta p}{k}+1\right)\right]^{1/p}}\left(\frac{1}{4}\right)^{1/q}\\
&\times\left(\left|\tfrac{\partial^2 g}{\partial s\partial t}(a_1,b_1)\right|^q+\left|\tfrac{\partial^2 g}{\partial s\partial t}(a_1,b_2)\right|^q+\left|\tfrac{\partial^2 g}{\partial s\partial t}(a_2,b_1)\right|^q+\left|\tfrac{\partial^2 g}{\partial s\partial t}(a_2,b_2)\right|^q\right)^{1/q}
\end{aligned}$$

where A_2 *is defined by (3.9).*

4. Conclusion

The authors predict that by using this new double generalized integral they have defined, new Hermite-Hadamard type inequalities will be obtained for different convex functions.The authors also hope that this work can stimulate further research into the various known types of inequalities and their applications of fractional integrals.

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