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# TAUBERIAN THEOREMS FOR FUZZY SEQUENCES BY WEIGHTED SUMMABILITY METHOD

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Abstract. Fuzzy set theory has entered into a large variety of disciplines of sciences, technology and humanities having established itself as an extremely versatile interdisciplinary research area. Accordingly different notions of fuzzy structure have been developed such as fuzzy normed linear space, fuzzy topological vector space, fuzzy sequence space etc. While reviewing the literature in fuzzy sequence space, we have seen that the notion of Tauberian theorems for the Nörlund-Cesáro mean-convergent sequences of fuzzy numbers has not been developed. In the present paper, we introduce some new concepts about statistical convergence of sequences of fuzzy numbers. The main purpose of this paper is to study Tauberian theorems for the Nörlund-Cesáro mean-convergent sequences of fuzzy numbers and investigate some other kind of convergences named Nörlund-Cesáro mean-level convergence so as to fill up the existing gaps in the literature. The results which we obtained in this study are much more general than those obtained by others.

## 1. introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [21] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [15] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. In addition, sequences of fuzzy numbers have been discussed by Kwon [12], Altin et al. [1], Gokhan et al. [10], Et et al. [7] and many others. The notion of statistical convergence was introduced by Fast [8] and Steinhaus [19], independently.

We shall denote by $\mathbb{N}$ the set of all natural numbers. Let $K \in \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of $K$ is defined by $d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to $L$ if for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero (cf. [8, 19]), i.e. for each $\epsilon > 0$,

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$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n:|x_k-L|\geq\epsilon\}\right|=0.$$

In this case, we write $L = st - \lim x$. Note that every convergent sequence is statistically convergent but not conversely.

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums $(s_n)$. The $(C,1)$ transform is defined as the n-th partial sum of $(C,1)$ summability and is given by

$$\frac{s_0+s_1+\cdots+s_n}{n+1}=\frac{1}{n+1}\sum_{k=0}^{n}s_k\to s \quad \text{as} \quad n\to\infty, \tag{1.1}$$

then the infinite series $\sum_{n=0}^{\infty} a_n$ is summable to the definite number $s$ by $(C,1)$ method.

Let $\{p_n\}$ be a non-negative, non increasing sequence such that

$$P_n=p_0+p_1+\cdots+p_n\to\infty, \quad \text{as} \quad n\to\infty, P_{-1}=p_{-1}=0, P_n\geq n.$$

Then the series $\sum_{n=0}^{\infty} a_n$ is said to be almost Nörlund summable to $S$ (or $(N,p_n)$-summable) if

$$\frac{1}{P_n}\sum_{v=0}^{n}p_{n-v}s_v\to S,$$

as $n\to\infty$.

The product of $(N,p_n)$ summability and $(C,1)$ summability defines $(N,p_n)(C,1)$ summability and we denote it by $N_n^pC_n^1$. Thus if

$$N_n^pC_n^1=\frac{1}{P_n}\sum_{k=0}^{n}p_kC_k^1\to s \quad \text{as} \quad n\to\infty, \tag{1.2}$$

where $N_n^p$ denotes the $(N,p_n)$ transform of $s_n$ and $C_n^1$ denotes the $(C,1)$ transform of $s_n$, then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable by $(N,p_n)(C,1)$ means or summable $(N,p_n)(C,1)$ to a definite number $s$. The $(N,p_n)$ is a regular method of summability(see [11]).

$$s_n\to s\Rightarrow C_n^1(s_n)=\frac{1}{n+1}\sum_{k=0}^{n}s_k\to s, \quad \text{as} \quad n\to\infty,$$

$C_n^1$ method is regular,

$$N_n^p(C_n^1(s_n))=N_n^pC_n^1\to s, \quad \text{as} \quad n\to\infty,$$

$N_n^pC_n^1$ method is regular.

We say that the sequence $(x_n)$ is Nörlund-Cesáro summable to $L$ by the weighted mean method determined by the sequences $(p_n)$, or briefly $(N,p_n)(C,1)-$ summable if

$$\lim_n N_n^pC_n^1(x)=L. \tag{1.3}$$

In this case we will write $L=N_n^pC_n^1-\lim_n x_n$. We denote by $N_n^pC_n^1$ the set of all sequences which are summable $N_n^pC_n^1$. If

$$\lim_n x_n=a \tag{1.4}$$

exists, then (2.1) also exists. However, the converse is not always true. We can show by the following example

**Example 1.1.** *Let us consider that* $p_n = 1$ *for all* $n \in \mathbb{N}$. *Also we define the following sequence* $x = (x_k) = (-1)^k$, *then we have*

$$\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+1}\sum_{v=0}^{k}(-1)^v \to 0 \quad as \quad n \to \infty.$$

*And as we know* $x = (x_k)$, *is not convergent.*

Notice that (2.1) may imply (2.2) under a certain condition, which is called a Tauberian condition. Any theorem which states that convergence of sequences follows from its $(N, p_n)(C, 1)-$ summability and some Tauberian condition is said to be a Tauberian theorem for the $(N, p_n)(C, 1)-$ summability method.

The theory of Tauberian is extensively studied by many authors([4], [6], [14],[16, 17], [22]). In this paper our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions convergence of sequences $(x_n)$, follows from $N_n^p C_n^1-$ convergence. This is a generalization of methods given in [17] and [22], and it is shown in the following example.

**Example 1.2.** *Let us consider that* $x_n = n$, *then* $N_n^p C_n^1$ *reduces to the Nörlund method defined in* [17] *and* [22].

**Definition 1.** *A sequence* $(x_n)$ *is weighted* $N_n^p C_n^1-$*statistically convergent to* $L$ *if for every* $\epsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{P_n}\left|\left\{k \le P_n : \left|\frac{1}{P_n}\sum_{k=0}^{n}p_k\frac{1}{k+1}\sum_{v=0}^{k}x_v - L\right| \ge \epsilon\right\}\right| = 0.$$

In paper [5], was given conditions under which for every bounded sequence $(x_k)$ the implication

$$st_\lambda - \lim_k x_k = L \quad \text{implies} \quad st_\lambda - \lim_k (EC)_k^1 = L$$

holds. The converse of the above fact is known as Tauberian Theorem. Also this fact, for the above summability method, is given in [5]. The theory of Tauberian theorems are intensively investigated by several authors, see [16], [13], [4], [6]. In this paper we will prove Tauberian theorems for fuzzy sequence spaces. Let $C(\mathbb{R})$ denote the family of all nonempty, compact, convex subsets of $\mathbb{R}$. Denote by

$$L(\mathbb{R}) = \{u : \mathbb{R} \to [0,1] : u \quad \text{satisfies} \quad (1) - (4) \quad \text{bellow}\}$$

where

(1) $u$ is normal, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,

(2) $u$ is fuzzy convex, for any $x, y \in \mathbb{R}$ and $0 \le \lambda \le 1$, $u(\lambda x + (1-\lambda)y) \ge \min[u(x), u(y)]$,

(3) $u$ is upper semicontinuous,

(4) the closure of $\{x \in \mathbb{R} : u(x) > 0\}$, denoted by $[u]_0$, is compact.

If $u \in L(\mathbb{R})$, then $u$ is called fuzzy number, and $L(\mathbb{R})$ is said to be fuzzy number space. For $0 < \alpha \le 1$, the $\alpha-$ level set $[u]_\alpha$ of $u$ is defined by $[u]_\alpha = \{x \in \mathbb{R} :$

$u(x) \geq \alpha\}$. Then from (1)-(4), it follows that the $\alpha-$level sets $[u]_\alpha \in C(\mathbb{R})$.

The set of real numbers can be embedded in $L(\mathbb{R})$, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r} = \begin{cases} 1; & \text{if} \quad x = r, \\ 0; & \text{if} \quad x \neq r. \end{cases}$$

Let $u, v, w \in L(\mathbb{R})$ and $k \in \mathbb{R}$. Then the operations addition and scalar multiplications are defined in $L(\mathbb{R})$ as follows:

$$u + v = w \Leftrightarrow [w]_\alpha = [u]_\alpha + [v]_\alpha \quad \text{for all} \quad \alpha \in [0,1],$$

$$\Leftrightarrow w_\alpha^- = u_\alpha^- + v_\alpha^- \quad \text{and} \quad w_\alpha^+ = u_\alpha^+ + v_\alpha^+ \quad \text{for all} \quad \alpha \in [0,1],$$

$$[ku]_\alpha = k[u]_\alpha \quad \text{for all} \quad \alpha \in [0,1].$$

Further details related to the structural properties of the fuzzy numbers, are given in [3]. Let us denote by $W$ the set of all closed bounded intervals $A$ of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e., $A = [\underline{A}, \overline{A}]$. Define the relation $d$ on $W$ by

$$d(A, B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can be easily observed that $d$ is a metric on $W$ and $(W, d)$ is a complete metric space, ([18]). Now, we may define the metric $D$ on $L(\mathbb{R})$ by means of the Hausdorff metric $d$ as follows

$$D(u, v) = \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) = \sup_{\alpha \in [0;1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\}.$$

And

$$D(u, 0) = \sup_{\alpha \in [0;1]} \max\{|u^-(\alpha)|, |u^+(\alpha)|\} = \max\{|u^-(\alpha)|, |u^+(\alpha)|\}.$$

A sequence $u = (u_k)$ of fuzzy numbers is a function $u$ from the set $\mathbb{N}$, into the set $L(\mathbb{R})$. The fuzzy number $u_k$ denotes the value of the function at $k \in \mathbb{N}$ and is called as the $k-$th term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers. A sequence $(u_n) \in w(F)$ is said to be convergent to $u \in L(\mathbb{R})$, if for every $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$D(u_n, u) < \epsilon \quad \text{for all} \quad n > n_0.$$

**Definition 2.** *Let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence $X$ is said to converge weighted statistically to a fuzzy number $X_0$, if for every $\epsilon > 0$*

$$\lim_{n \to \infty} \frac{1}{P_n} |\{k \leq P_n : D(X_k, X_0) \geq \epsilon\}| = 0.$$

The above type of convergence we will denote by

$$st_{FW} - \lim_n X_n = X_0.$$

**Definition 3.** *Let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence $X$ is said to be Nörlund-Cesáro summable to a fuzzy number $X_0$ if the sequence*

$$N_n^p C_n^1(X) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} X_v,$$

*is statistically convergent to* $X_0$*, where the sum in* $N_n^p C_n^1(X)$ *is usual addition of fuzzy real numbers through* $\alpha-$ *level sets. That is* $(X_k)$ *is statistically Nörlund-Cesáro summable to the fuzzy number* $X_0$*, if for every* $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n} \left|\{k \le P_n : D(N_k^p C_k^1, X_0) \ge \epsilon\}\right| = 0.$$

The above type of convergence we will denote by

$$st_F - \lim_n N_n^p C_n^1 = X.$$

**Remark.** *From statistically Nörlund-Cesáro summable, follows statistically Nörlund-Cesáro summable of the fuzzy sequences.*

**Theorem 1.3.** *Let us suppose that* $(X_k)$ *is a bounded sequence of fuzzy numbers such that exists* $st_{FW} - \lim_k X_k = L$*, then it follows that* $st_F - \lim_k N_n^p C_n^1 = L$*, but not conversely.*

*Proof.* Let us suppose that $st_{FW} - \lim_k X_k = L$. Let $\epsilon > 0$ be any given number. Then

$$D(N_n^p C_n^1, L) = D\left(\frac{1}{P_n}\sum_{k=0}^{n}\frac{1}{k+1}\sum_{v=0}^{k} X_v, L\right) = D\left(\frac{1}{P_n}\sum_{k=0}^{n}\frac{1}{k+1}\sum_{v=0}^{k} X_v, \frac{P_n}{P_n}L\right)$$

$$= \left|\frac{1}{P_n}\right| D\left(\sum_{k=0}^{n}\frac{1}{k+1}\sum_{v=0}^{k} X_v, P_n \cdot L\right) \le \frac{1}{P_n} D\left(\sum_{k=0}^{n}\frac{1}{k+1}\sum_{v=0}^{k} X_v, L\right)$$

$$\frac{1}{P_n} D\left(\frac{X_0}{1} + \frac{X_0 + X_1}{2} + \cdots + \frac{X_0 + X_1 + X_2 + \cdots + X_n}{n+1}, L\right) \le$$

$$\frac{1}{P_n} D\left(\frac{X_0 + X_1 + X_2 + \cdots + X_n}{n+1}, L\right) \le$$

$$\frac{1}{(n+1)P_n} D\left(X_0 + X_1 + X_2 + \cdots + X_n, (n+1)L\right) \le$$

$$\frac{1}{(n+1)P_n}\left[D(X_0, L) + D(X_1, L) + D(X_2, L) + \cdots + D(X_n, L)\right] \le$$

$$\le \frac{1}{P_n} \sup_{j\in\{0,1,2,\cdots,n\}} \{D(X_j, L)\} \le$$

$$\frac{1}{P_n} \sup_{j\le P_n} \{D(X_j, L)\} \to 0, \quad \text{as} \quad n\to\infty.$$

To prove that converse is not true, we construct this example

**Example 1.4.** *Let us consider that* $(p_n) = n$ *and the following sequence* $(X_k)$ *of fuzzy numbers:*

$$X_k(t) = \begin{cases} \left.\begin{array}{cc} t+1, & -1 \le t \le 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & \textit{otherwise} \end{array}\right\}, & \textit{for } k = m^2 - m, ..., m^2 - 1 \\ \left.\begin{array}{cc} 1 - \frac{t}{m}, & -m \le t \le 0 \\ 1 + \frac{t}{m}, & 0 \le t \le m \\ 0, & \textit{otherwise} \end{array}\right\}, & \textit{for } k = m^2, m = 2, ... \\ \left.\begin{array}{cc} 0, & \textit{otherwise} \end{array}\right\}, & \textit{otherwise} \end{cases}$$

*Then, we calculate the $\alpha-$level sets of sequences $(X_k)$ as follows*

$$[X_k]^\alpha = \begin{cases} [\alpha-1, 1-\alpha] & for \quad k = m^2-m, ..., m^2-1 \\ [-m(\alpha-1), -m(1-\alpha)] & k = m^2; m = 2, 3, \cdots \\ \hat{0} & otherwise \end{cases}$$

*The sequence of fuzzy numbers $(X_k)$ is $N_n^pC_n^1$ summable to $\hat{0}$, because*

$$N_n^pC_n^1 = \begin{cases} \frac{2(l+1)(\ln n + C + \epsilon_n)}{n(n+1)} & for \quad k = m^2-m+l; l = 0, 1, 2, \cdots, m-1; m = 2, 3, \\ \hat{0} & otherwise \end{cases},$$

*where $C$ is a Euler constant and $\epsilon_n \to 0$, as $n \to \infty$. And hence statistically $N_n^pC_n^1-$ summable. On the other hand, the sequence $(m^2; m = 2, 3\cdots,)$ is statistically convergent to 0, it is clear that $st - \liminf_n X_n = 0$ and $st - \limsup_n X_n = 1$. Thus $X = (X_n)$ is not statistically convergent. Nor weighted statistically convergent.*

□

In this paper our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions weighted statistically convergence follows from $N_n^pC_n^1$-statistically convergence.

## 2. MAIN RESULTS

**Theorem 2.1.** *Let us consider that*

$$st_{FW} - \liminf_n \frac{P_{t_n}}{P_n} > 1, \quad t > 1 \tag{2.1}$$

*where $t_n$, denotes the integer part of the $[t \cdot n]$ for every $n \in \mathbb{N}$, and let $N_n^pC_n^1$ be a sequence of fuzzy real numbers such that $st_F - \lim_k N_n^pC_n^1 = L$. Then $(X_k)$ is $st_{FW}-$ convergent to the same fuzzy number $L$ if and only if the following to conditions holds:*

$$\inf_{t>1} \limsup_n \frac{1}{P_n} \left| \left\{ k \le P_n : D\left( \frac{1}{P_{t_k} - P_k} \sum_{j=k+1}^{t_k} p_j \frac{1}{j+1} \sum_{v=0}^{j} (X_v - X_k), X_k \right) \ge \epsilon \right\} \right| = 0, \tag{2.2}$$

*and*

$$\inf_{0<t<1} \limsup_n \frac{1}{P_n} \left| \left\{ k \le P_n : D\left( \frac{1}{P_k - P_{t_k}} \sum_{j=t_k+1}^{k} p_j \frac{1}{j+1} \sum_{v=0}^{j} (X_k - X_v), X_k \right) \ge \epsilon \right\} \right| = 0. \tag{2.3}$$

**Remark.** *Let us suppose that $st_{FW} - \lim_k x_k = L$; $st_F - \lim_k N_n^pC_n^1 = L$ and relation (2.1) satisfies, then for every $t > 1$, is valid the following relation:*

$$st_{FW} - \lim_n \frac{1}{P_{t_k} - P_k} \sum_{j=k+1}^{t_k} p_j \frac{1}{j+1} \sum_{v=0}^{j} X_v = 0, \tag{2.4}$$

*and in case where $0 < t < 1$,*

$$st_{FW} - \lim_n \frac{1}{P_k - P_{t_k}} \sum_{j=t_k+1}^{k} p_j \frac{1}{j+1} \sum_{v=0}^{j} X_v = 0. \tag{2.5}$$

In what follows we will show some auxiliary lemmas which are needful in the sequel.

**Lemma 2.2.** *The condition given by relation (2.1) is equivalent to this one:*

$$st_{FW}-\liminf_{n}\frac{P_n}{P_{t_n}}>1,\quad 0<t<1. \tag{2.6}$$

*Proof.* Let us suppose that relation (2.1) is valid, $0<t<1$ and $m=t_n=[t\cdot n]$, $n\in\mathbb{N}$. Then it follows that

$$\frac{1}{t}>1\Rightarrow\frac{m}{t}=\frac{[t\cdot n]}{t}\leq n,$$

we have:

$$\frac{P_n}{P_{t_n}}\geq\frac{P_{\left[\frac{m}{t}\right]}}{P_m}\Rightarrow st_{FW}-\liminf_{n}\frac{P_n}{P_{t_n}}\geq st_{FW}-\liminf_{n}\frac{P_{\left[\frac{m}{t}\right]}}{P_{t_n}}>1.$$

Conversely, let use suppose that relation (2.6) is valid. Let $t>1$ and be given and let $t_1$ be chosen such that $1<t_1<t$. Set $m=t_n=[t\cdot n]$. From $0<\frac{1}{t}<\frac{1}{t_1}<1$, it follows that:

$$n\leq\frac{tn-1}{t_1}<\frac{[tn]}{t_1}=\frac{m}{t_1},$$

provided $t_1\leq t-\frac{1}{n}$, which is a case where if $n$ is large enough.Under this conditions we have:

$$\frac{P_{t_n}}{P_n}\geq\frac{P_{t_n}}{P_{\left[\frac{m}{t_1}\right]}}\Rightarrow st_{FW}-\liminf_{n}\frac{P_{t_n}}{P_n}\geq st_{FW}-\liminf_{n}\frac{P_{t_n}}{P_{\left[\frac{m}{t_1}\right]}}>1.$$

□

**Lemma 2.3.** *Let $(p_n)$ be a sequence which is non-negative sequence of numbers and satisfies condition (2.1). If $X=(X_k)$ is a sequence of fuzzy numbers converges $N_n^pC_n^1-$statistically to fuzzy number $L$. Then for every $t>0$,*

$$st_F-\lim_n N_{t_n}^pC_n^1=L,$$

*where by $t_n=[t\cdot n]$, is denote the integer part of the product $t\cdot n$.*

*Proof.* Proof of the Lemma is similar to the proof of Lemma 2, in [17]. For this reason we omit it. □

**Lemma 2.4.** *Let $(p_n)$ be a sequence which is non-negative sequence of numbers and satisfies condition (2.1).If $X=(X_k)$ is a sequence of fuzzy numbers which converges $N_n^pC_n^1-$statistically to fuzzy number $L$. Then for every $t>1$,*

$$st_{FW}-\lim_n\left(P_{t_n}-P_n\right)^{-1}\sum_{j=n+1}^{t_n}p_j\frac{1}{j+1}\sum_{v=0}^{j}X_j=L; \tag{2.7}$$

*and for every $0<t<1$,*

$$st_{FW}-\lim_n\left(P_n-P_{t_n}\right)^{-1}\sum_{j=t_n+1}^{n}p_j\frac{1}{j+1}\sum_{v=0}^{j}X_j=L. \tag{2.8}$$

*Proof.* Let us suppose that $t > 1$, then $P_{t_n} > P_n$, and

$$D\left(\frac{1}{P_{t_n} - P_n}\sum_{k=n+1}^{t_n} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v, L\right) \leq$$

$$D\left(\frac{1}{P_{t_n} - P_n}\sum_{k=n+1}^{t_n} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v, N_n^p C_n^1\right) + D\left(N_n^p C_n^1, L\right). \tag{2.9}$$

It is know that for any three fuzzy numbers $X, Y$ and $Z$, is valid the following relation:

$$D(X+Y, Z) \leq D(X, Z). \tag{2.10}$$

(I) Let us suppose that $t > 1$. Knowing that $D(x, y) = D(x+t, y+t)$, then after some calculations we obtain

$$D\left(\frac{1}{P_{t_n} - P_n}\sum_{k=n+1}^{t_n} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v, N_n^p C_n^1\right) =$$

$$D\left(\frac{P_{t_n}}{P_{t_n} - P_n} N_{t_n}^p C_n^1, N_n^p C_n^1 + \frac{1}{P_{t_n} - P_n}\sum_{k=0}^{n} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v\right)$$

$$\leq D\left(\frac{P_{t_n}}{P_{t_n} - P_n} N_{t_n}^p C_n^1, N_n^p C_n^1\right) = D\left(\frac{P_{t_n}}{P_{t_n} - P_n} N_{t_n}^p C_n^1, \frac{P_{t_n}}{P_{t_n}} N_n^p C_n^1\right) =$$

$$\frac{P_{t_n}}{P_{t_n} - P_n} D\left(N_{t_n}^p C_n^1, \left(1 - \frac{P_n}{P_{t_n}}\right) N_n^p C_n^1\right) \leq$$

$$\frac{P_{t_n}}{P_{t_n} - P_n} D\left(N_{t_n}^p C_n^1, N_n^p C_n^1\right). \tag{2.11}$$

From definition of the sequence $(t_n)$, we get

$$st_{FW} - \limsup_{n} \frac{P_{t_n}}{P_{t_n} - P_n} < \infty. \tag{2.12}$$

Now relation (2.7) follows from relations (2.11), (2.12) and Lemma 2.3.

(II) The case $0 < t < 1$. In this case we have

$$D\left(\frac{1}{P_n - P_{t_n}}\sum_{k=t_n+1}^{n} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v, L\right) \leq$$

$$D\left(\frac{1}{P_n - P_{t_n}}\sum_{k=t_n+1}^{n} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v, N_{t_n}^p C_n^1\right) + D\left(N_{t_n}^p C_n^1, L\right).$$

Following Lemma 2.3 and the conclusions like as in the previous case we get that relation (2.8) is valid. □

In what follows we will prove the Theorem 2.1.

*Proof.* Necessity. Let us suppose that $st_{FW} - \lim_k X_k = X_0$, and $st_F - \lim_k N_n^p C_k^1 = X_0$. For every $t$ following Lemma 2.3 we get relation (2.2) and relation (2.3).

Sufficient: Assume that $st_F - \lim_n N_n^p C_n^1 = X_0$, and conditions (2.1), (2.2) and (2.3) are satisfied. In what follows we will prove that $st_{FW} - \lim_n X_n = X_0$. Or

equivalently, $st_F - \lim_n D(N_n^p C_n^1, X_n) = 0$.
First we consider the case where $t > 1$. Since

$$D(N_n^p C_n^1, X_n) \leq D\left(\frac{1}{P_{t_n} - P_n}\sum_{k=n+1}^{t_n} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v, N_n^p C_n^1\right) +$$

$$D\left(\frac{1}{P_{t_n} - P_n}\sum_{k=n+1}^{t_n} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v, X_n\right).$$

For any $\epsilon > 0$, we get:

$$\{k \leq P_n : D(X_k, N_k^p C_k^1) \geq \epsilon\} \subset$$

$$\left\{k \leq P_n : D\left(\frac{1}{P_{t_k} - P_k}\sum_{j=k+1}^{t_k} p_j \frac{1}{j+1}\sum_{v=0}^{j} X_v, X_k\right) \geq \epsilon\right\}$$

$$\cup \left\{k \leq P_n : \frac{P_{t_k}}{P_{t_k} - P_k} \cdot D\left(N_{t_k}^p C_k^1; N_k^p C_k^1\right) \geq \frac{\epsilon}{2}\right\}. \tag{2.13}$$

From relation (2.2), it follows that for every $\delta > 0$, exists a $t > 1$ such that

$$\limsup_n \left|\left\{k \leq P_n : D\left(\frac{1}{P_{t_k} - P_k}\sum_{j=k+1}^{t_k} p_j \frac{1}{j+1}\sum_{v=0}^{j} X_v, X_k\right) \geq \epsilon\right\}\right| \leq \delta. \tag{2.14}$$

On the other side, from Lemma 2.3, it follows that

$$\lim_{n\to\infty} \left|\left\{k \leq P_n : \frac{P_{t_k}}{P_{t_k} - P_k} \cdot D\left(N_{t_k}^p C_k^1; N_k^p C_k^1\right) \geq \frac{\epsilon}{2}\right\}\right| = 0. \tag{2.15}$$

From relations (2.13), (2.14), (2.15) and (2.1), it yields

$$\lim_{n\to\infty} \left|\left\{k \leq P_n : D(N_k^p C_k^1; X_k) \geq \epsilon\right\}\right| \leq \delta.$$

Since $\delta > 0$ is arbitrary , we can conclude that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \left|\left\{k \leq P_n : D(N_k^p C_k^1; X_k) \geq \epsilon\right\}\right| = 0.$$

Now, in similar way as above we can prove the second case.

□

## 3. Tauberian theorems for Nörlund-Cesáro mean-level convergence

In this section we will introduce some level convergence of fuzzy numbers which is generalization of that given by [20] and also we will give some results related to the statistical limit inferior and superior for sequence of fuzzy numbers related to the new definition.

Let $X = (X_n)$ and $Y = (Y_n)$ be two real sequences of fuzzy numbers. We say that $(X_k)$ is weighted statistically bounded from above if there exists a fuzzy number $X_0$, such that

$$\lim_n \frac{1}{P_n} \left|\{k \leq P_n : X_k > X_0\} \cup \{k \leq P_n : X_k \nsim X_0\}\right| = 0.$$

Similarly, $(X_k)$ is weighted statistically bounded from bellow if there exists a fuzzy number $Y_0$, such that

$$\lim_n \frac{1}{P_n} |\{k \leq P_n : X_k < Y_0\} \cup \{k \leq P_n : X_k \nsim Y_0\}| = 0.$$

If the sequence $X = (X_k)$ is both weighted statistically bounded from above and bellow we will say that it is weighted statistically bounded.

With respect to order-theoretical aspect, there are several possibilities to define an order relation on the set of fuzzy numbers. As it is customary, we will basically follow the order relation on $L(\mathbb{R})$ defined by $X \leq Y$ if and only if $X^-(\alpha) \leq Y^-(\alpha)$ and $X^+(\alpha) \leq Y^+(\alpha)$, for any $\alpha \in [0,1]$. Where $[X]_\alpha = [X^-(\alpha), X^+(\alpha)]$. The fuzzy numbers $X$ and $Y$ are said to be incomparable if neither $X \leq Y$ nor $Y \leq X$ holds. In this case, we use the notation $X \nsim Y$.

**Definition 4.** *Let $X = (X_n)$ be a weighted statistically bounded sequence of fuzzy numbers. Then the weighted statistical limit inferior of the $X = (X_n)$ is given by*

$$st_{FW} - \liminf_n X_n = \inf \left\{ X_0 \in E : \lim_n \frac{1}{P_n} |k \leq P_n : X_k < X_0| \neq 0 \right\}.$$

*And, the weighted statistical limit superior of the $X = (X_n)$ is given by*

$$st_{FW} - \limsup_n X_n = \inf \left\{ X_0 \in E : \lim_n \frac{1}{P_n} |k \leq P_n : X_k > X_0| \neq 0 \right\}.$$

**Theorem 3.1.** *Let $X = (X_n)$ be a weighted statistically bounded sequence of fuzzy numbers. If $st_{FW} - \lim_n \inf X_n = X_0$, then*

$$\lim_n \frac{1}{P_n} |\{k \leq P_n : X_k < X_0 - \epsilon\}| = 0 \tag{3.1}$$

*and*

$$\lim_n \frac{1}{P_n} |\{k \leq P_n : X_k < X_0 + \epsilon\} \cup \{k \leq P_n : X_k \nsim X_0 + \epsilon\}| \neq 0, \tag{3.2}$$

*for every $\epsilon > 0$.*

We will follows techniques given in Theorem 2([2]), to prove our theorem.

*Proof.* Let us suppose the contrary, that there exists a $\epsilon > 0$ such that

$$\frac{1}{P_n} |\{k \leq P_n : X_k < X_0 - \epsilon\}| \nrightarrow 0,$$

as $n \to \infty$. From definition 4 and above relation we obtain that $X_0 < X_0 - \epsilon$, which is contradiction. In the second case, let us suppose again contrary, there exists $\epsilon > 0$ such that

$$\frac{1}{P_n} |\{k \leq P_n : X_k < X_0 + \epsilon\}| \to 0, \quad \frac{1}{P_n} |\{k \leq P_n : X_k \nsim X_0 + \epsilon\}| \to 0, \tag{3.3}$$

as $n \to \infty$. For each $k \leq P_n$, for these two fuzzy numbers $X_k$ and $X_0 + \epsilon$, we have this three possibilities:

$X_k < X_0 + \epsilon$, $X_k > X_0 + \epsilon$ and $X_k \nsim X_0 + \epsilon$. And we get

$$\{k \leq P_n : X_k < X_0 + \epsilon\} \cup \{k \leq P_n : X_k > X_0 + \epsilon\} \cup \{k \leq P_n : X_k \nsim X_0 + \epsilon\} = P_n. \tag{3.4}$$

Comparing relations (3.3) and (3.4), we obtain

$$\lim_n \frac{|\{k \leq P_n : X_k > X_0 + \epsilon\}|}{P_n} = 1.$$

This means that $X_0 + \epsilon < X_0$, which is contradiction. And this proves theorem. $\square$

The same result is valid for the statistically limit superior, as is shown in the following

**Theorem 3.2.** *Let $X = (X_n)$ be a statistically bounded sequences of fuzzy numbers. If $st_{FW} - \lim_n \sup X_n = X_0$, then*

$$\lim_n \frac{1}{P_n} |\{k \leq P_n : X_k > X_0 + \epsilon\}| = 0 \tag{3.5}$$

*and*

$$\lim_n \frac{1}{P_n} |\{k \leq P_n : X_k > X_0 - \epsilon\} \cup \{k \leq P_n : X_k \nsim X_0 - \epsilon\}| \neq 0, \tag{3.6}$$

*for every $\epsilon > 0$.*

The level convergence of sequence of fuzzy numbers was given by [9], as follows. Let $X = (X_k)$ be a sequence of fuzzy numbers, it is level-convergent to the fuzzy number $X_0$, if

$$\lim_{n\to\infty} X_n^-(\alpha) = X_0^-(\alpha), \lim_{n\to\infty} X_n^+(\alpha) = X_0^+(\alpha), \tag{3.7}$$

for any $\alpha \in [0,1]$.

**Remark.** *The level convergence differs from the standard convergence and it is shown in the following example.*

**Example 3.3.** *Let*

$$\underline{u_n}(r) = \begin{cases} (r - \frac{1}{4})^n; & \frac{1}{4} < r \leq 1 \\ 0; & 0 \leq r \leq \frac{1}{4} \end{cases}, \quad \overline{u_n}(r) = 1,$$

*and*

$$\underline{u_0}(r) = \begin{cases} 1; & \frac{1}{4} < r \leq 1 \\ 0; & 0 \leq r \leq \frac{1}{4} \end{cases}, \quad \overline{u_0}(r) = 1.$$

*From representation theorem in [20], there exists a unique fuzzy number $u_n$ and a unique fuzzy number $u_0$ such that $[u_n]^r = [\underline{u_n}(r), \overline{u_n}(r)], [u_0]^r = [\underline{u_0}(r), \overline{u_0}(r)]$. Obviously, the sequences $\underline{u_n}(r), \overline{u_n}(r)$ converges to $\underline{u_0}(r), \overline{u_0}(r)$, respectively at any $r \in [0,1]$, as $n \to \infty$. And $D(u_n, u_0) = \sup_{r\in(\frac{1}{4},1]} \left\{1 - (r - \frac{1}{4})^{\frac{1}{n}}\right\} = 1$, for any natural number $n$. But $u_n$ does not converges to $u_0$.*

Following it and our definition related to the Euler-Nörlund mean-convergent, we give this

**Definition 5.** *The fuzzy sequence $X = (X_k)$ is Nörlund-Cesáro mean-level convergent to a fuzzy number $X_0$, if*

$$\lim_{n\to\infty} \frac{1}{P_n} \sum_{k=1}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} X_k^-(\alpha) = X_0^-(\alpha), \lim_{n\to\infty} \frac{1}{P_n} \sum_{k=1}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} X_k^+(\alpha) = X_0^+(\alpha),$$

*for all $\alpha \in [0,1]$.*

In what follows we will prove a kind of Tauberian theorem for the newly defined concept of Nörlund-Cesáro mean-level convergent.

**Theorem 3.4.** *Let $X = (X_k)$ be a bounded sequence of fuzzy numbers. Assume that $(X_k)$ is Nörlund-Cesáro mean-level convergent to fuzzy number $X_0$. Also we suppose that $st_{FW} - \lim_n \sup X_k = X_0$ and there is a number $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$,*

$$\lim_n \frac{1}{P_n} \left| \left\{ k \leq P_n : p_k \frac{1}{k+1} \sum_{v=0}^{k} X_v \nsim X_0 - \epsilon \right\} \right| = 0,$$

$$\lim_n \frac{1}{P_n} \left| \left\{ k \leq P_n : p_k \frac{1}{k+1} \sum_{v=0}^{k} X_v \nsim X_0 + \epsilon \right\} \right| = 0.$$

*Then $st_{FW} - lim_n X_n = X_0$.*

*Proof.* We say that fuzzy numbers $v$ and $w$ are not comparable if neither $v \leq w$ nor $w \leq v$. From $st_{FW} - \lim_n \sup X_k = X_0$ and Theorem 3.2, we have

$$\lim_n \frac{1}{P_n} \left| \{ k \leq P_n : X_k > X_0 + \epsilon \} \right| = 0,$$

for every $\epsilon > 0$. Let us suppose for the moment that

$$\lim_n \frac{1}{P_n} \left| \{ k \leq P_n : D(X_k, X_0) > \epsilon \} \right| \neq 0.$$

Then exists a $\epsilon_1 \in (0, \epsilon_0)$ such that

$$\lim_n \frac{1}{P_n} \left| \{ k \leq P_n : X_k < X_0 - \epsilon_1 \} \right| \neq 0.$$

We define the following sets of fuzzy numbers:

$$A_1 = \left\{ k \leq P_n : p_k \frac{1}{k+1} \sum_{v=0}^{k} X_k < X_0 - \epsilon \right\},$$

$$A_2 = \left\{ k \leq P_n : X_0 - \epsilon < p_k \frac{1}{k+1} \sum_{v=0}^{k} X_k < X_0 + \epsilon \right\},$$

$$A_3 = \left\{ k \leq P_n : p_k \frac{1}{k+1} \sum_{v=0}^{k} X_k > X_0 + \epsilon \right\} \cup \left\{ k \leq P_n : p_k \frac{1}{k+1} \sum_{v=0}^{k} X_k \nsim X_0 - \epsilon \right\} \cup$$
$$\left\{ k \leq P_n : p_k \frac{1}{k+1} \sum_{v=0}^{k} X_k \nsim X_0 + \epsilon \right\}.$$

From definitions of the sets $A_i, i \in \{1, 2, 3\}$, we get

$$\lim_n \frac{1}{P_n} |A_3| = 0, \lim_n \frac{1}{P_n} |A_1| \neq 0, \lim_n \frac{1}{P_n} |A_2| = 1 - \lim_n \frac{1}{P_n} |A_1|.$$

On the other side, from second relation in the expression we have that

$$\frac{1}{P_n} |A_1| \geq a > 0,$$

for infinitely $n$ and after some calculations we obtain

$$N_n^p C_n^1 = \frac{1}{P_n}\sum_{k=1}^{n} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v = \frac{1}{P_n}\sum_{k\in A_1} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v + \frac{1}{P_n}\sum_{k\in A_2} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v +$$

$$\frac{1}{P_n}\sum_{k\in A_3} p_k \frac{1}{k+1}\sum_{v=0}^{k} X_v < \frac{X_0-\epsilon}{P_n}|A_1| + \frac{X_0+\epsilon}{P_n}|A_2| + \frac{A}{P_n}|A_3|,$$

for some fuzzy number $A$. There exists an $\alpha \in [0,1]$ such that

$$N_n^p {C_n^1}^-(\alpha) < X_0^-(\alpha) + \epsilon(1-a) + 0(1).$$

since $\epsilon \in (0, \epsilon_0)$ is arbitrary, then we get

$$\liminf N_n^p {C_n^1}^-(\alpha) \le X_0^-(\alpha).$$

Hence $X = (X_n)$ is not Nörlund-Cesáro mean-level convergent to fuzzy number $X_0$, which prove Theorem. □

**Remark.** *The above theorem is valid also for the* $st_{FW} - \liminf_n X_n = X_0$.

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