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A NEW GENERAL INEQUALITY FOR LIPSCHITZ FUNCTIONS VIA HADAMARD FRACTIONAL INTEGRAL

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Abstract. In this paper, a general inequality is obtained for Lipschitz functions via inequalities used for GA-convex functions. Through this general inequality, some new Hadamard-type, Ostrowski-type, Simpson-type and Bullen-type inequalities for Lipschitzian functions via Hadamard fractional integral are obtained in the special cases. Finally, some applications to special means of these inequalities are also given.

1. Preliminaries

Let real function f be defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Following inequalities are well known in the literature as Hermite-Hadamard inequality, Ostrowski inequality and Simpson inequality respectively:

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a,b]$, then we the following inequality holds*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right]$$

for all $x \in [a,b]$.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality*

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holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

Recently, Tseng et al. [21] have established the following Hadamard-type inequality which refines the inequality (1.2).

Theorem 1.4 ([21])**.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$. Then we have the inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\leq\frac{f(a)+f(b)}{2} \end{aligned} \tag{1.3}$$

The third inequality in (1.3) is known in the literature as the Bullen inequality

In recent years, many athors have studied errors estimations for Hermite-Hadamard, Ostrowski, Simpson and Bullen inequalities; for some refinements, counterparts, generalization see [1, 2, 5, 6, 10, 11, 12, 13, 18, 19, 20, 24, 25, 26] and references therein.

The following definitions are well known in the literature.

Definition 1. *A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is called an M-Lipschitzian function on the interval I of real numbers with $M\geq 0$, if*

$$|f(x)-f(y)|\leq M|x-y|$$

for all $x,y\in I$.

For some recent results connected with Hermite-Hadamard type integral inequalities for Lipschitzian functions, see [3, 4, 8, 9, 22, 23].

Definition 2 ([16, 17])**.** *A function $f:I\subseteq(0,\infty)\to\mathbb{R}$ is said to be GA-convex (geometric-arithmatically convex) if*

$$f(x^ty^{1-t})\leq tf(x)+(1-t)f(y)$$

for all $x,y\in I$ and $t\in[0,1]$.

For some recent results different type integral inequalities for GA-convex functions, see [7, 15, 24] and references therein.

We will now give definitions of the right-sided and left-sided Hadamard fractional integrals which are used throughout this paper.

Definition 3. *Let $f\in L[a,b]$. The right-sided and left-sided Hadamard fractional integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of oder $\alpha>0$ with $b>a\geq 0$ are defined by*

$$J^\alpha_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{\alpha-1}f(t)\frac{dt}{t},\ a<x<b$$

and

$$J^\alpha_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\ln\frac{t}{x}\right)^{\alpha-1}f(t)\frac{dt}{t},\ a<x<b$$

respectively, where $\Gamma(\alpha)$ *is the Gamma function defined by* $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$ *(* [14]*).*

In [7], Iscan established Hermite-Hadamard's inequalities for GA-convex functions in Hadamard fractional integral forms as follows.

Theorem 1.5. *Let* $f:I\subseteq(0,\infty)\to\mathbb{R}$ *be a function such that* $f\in L[a,b]$*, where* $a,b\in I$ *with* $a<b$*. If* f *is a GA-convex function on* $[a,b]$*, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right)\leq\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left\{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right\}\leq\frac{f(a)+f(b)}{2}\tag{1.4}$$

with $\alpha>0$*.*

In the inequality (1.4), if we take $\alpha=1$, then we have the following inequality

$$f\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt\leq\frac{f(a)+f(b)}{2}.\tag{1.5}$$

Morever in [7], Iscan obtained a generalization of Hadamard, Ostrowski and Simpson type inequalities for quasi-geometrically convex functions via Hadamard fractional integrals as related the inequality (1.4).

In this paper, the authors obtain a new general inequality for Lipschitzian functions via Hadamard fractional integrals as related the inequality (1.4). The importance of this general inequality is that some new Hadamard-type, Ostrowski-type, Simpson-type and Bullen-type inequalities for Lipschitzian functions via Hadamard fractional integral are obtained in the special cases of the main Theorem.

2. Hadamard-type inequalities for Lipschitzian functions

Let the function $f:\mathbb{R}^+\to\mathbb{R}$ be a M-Lipschitzian function on I. Throughout this section we will take

$$\begin{aligned}&L_f\left(x,y,z,\lambda,\eta,\mu,\alpha,a,b\right)\\ =\;&\lambda^{\alpha}f\left(x\right)+\eta^{\alpha}f\left(y\right)+\mu^{\alpha}f\left(z\right)-\frac{\Gamma\left(\alpha+1\right)}{\left[\ln\frac{b}{a}\right]^{\alpha}}\left[J_{G_1-}^{\alpha}f(a)+J_{G_1+}^{\alpha}f(G_2)+J_{G_2+}^{\alpha}f(b)\right]\end{aligned}$$

and

$$\begin{aligned}&G_{\alpha,\lambda,\eta,\mu}\left(x,y,z\right)\\ =\;&\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{|x-t|}{t}dt+\int_{G_1}^{G_2}\left[\ln\frac{G_2}{t}\right]^{\alpha-1}\frac{|y-t|}{t}dt+\int_{G_2}^{b}\left[\ln\frac{b}{t}\right]^{\alpha-1}\frac{|z-t|}{t}dt\end{aligned}$$

where $a,b\in I$ with $a<b$, $x,y,z\in[a,b]$, $\lambda,\eta,\mu\in[0,1]$, $G_1=a^{1-\lambda}b^{\lambda}$, $G_2=a^{\mu}b^{\lambda+\eta}$, $\alpha>0$ and $\Gamma\left(\alpha\right)=\int_0^\infty e^{-t}t^{\alpha-1}dt$ is Euler Gamma function.

Theorem 2.1. *Let the function* $f:I\subseteq\mathbb{R}^+\to\mathbb{R}$ *be a* M*-Lipschitzian function on* I *and assume that* $a,b\in I$ *with* $a<b$, $a\leq x\leq y\leq z\leq b$ *and* $\lambda+\eta+\mu=1$*;* $\lambda,\eta,\mu\in[0,1]$*. Then for all* $x,y,z\in[a,b]$ *and* $\alpha>0$ *we have the following inequality*

for Hadamard fractional integrals

$$\begin{aligned} & |L_f(x,y,z,\lambda,\eta,\mu,\alpha,a,b)| \\ \leq & \frac{\alpha M}{\left[\ln\frac{b}{a}\right]^{\alpha}}\left[\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{|x-t|}{t}dt+\int_{G_1}^{G_2}\left[\ln\frac{G_2}{t}\right]^{\alpha-1}\frac{|y-t|}{t}dt\right. \\ & \left.+\int_{G_2}^{b}\left[\ln\frac{b}{t}\right]^{\alpha-1}\frac{|z-t|}{t}dt\right] \\ = & \frac{\alpha M G_{\alpha,\lambda,\eta,\mu}(x,y,z)}{\left[\ln\frac{b}{a}\right]^{\alpha}}. \end{aligned}$$

Proof. Firstly, by changing the variable $x=\ln\frac{t}{a}$ in the identity

$$\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{f(u)}{t}dt=f(u)\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{dt}{t},$$

it is easily seen that

$$\begin{aligned} f(u)\int_0^{\lambda\ln\frac{b}{a}}x^{\alpha-1}dx &= \frac{f(u)}{\alpha}\lambda^{\alpha}\left[\ln\frac{b}{a}\right]^{\alpha} \\ \lambda^{\alpha}f(x) &= \frac{\alpha}{\left[\ln\frac{b}{a}\right]^{\alpha}}\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{f(t)}{t}dt. \end{aligned}$$

Similarly, we get

$$\eta^{\alpha}f(y)=\frac{\alpha}{\left[\ln\frac{b}{a}\right]^{\alpha}}\int_{G_1}^{G_2}\left[\ln\frac{G_2}{t}\right]^{\alpha-1}\frac{f(t)}{t}dt$$

and

$$\mu^{\alpha}f(z)=\frac{\alpha}{\left[\ln\frac{b}{a}\right]^{\alpha}}\int_{G_2}^{b}\left[\ln\frac{b}{t}\right]^{\alpha-1}\frac{f(t)}{t}dt.$$

Therefore, we have the following inequality:

$$\begin{aligned}
& |L_f(x,y,z,\lambda,\eta,\mu,\alpha,a,b)| \\
= & \frac{\alpha}{\left[\ln\frac{b}{a}\right]^{\alpha}}\left|\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{f(x)-f(t)}{t}dt+\int_{G_1}^{G_2}\left[\ln\frac{G_2}{t}\right]^{\alpha-1}\frac{f(y)-f(t)}{t}dt\right. \\
& \left.+\int_{G_2}^{b}\left[\ln\frac{b}{t}\right]^{\alpha-1}\frac{f(z)-f(t)}{t}dt\right| \\
\leq & \frac{\alpha}{\left[\ln\frac{b}{a}\right]^{\alpha}}\left[\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{|f(x)-f(t)|}{t}dt+\int_{G_1}^{G_2}\left[\ln\frac{G_2}{t}\right]^{\alpha-1}\frac{|f(y)-f(t)|}{t}dt\right. \\
& \left.+\int_{G_2}^{b}\left[\ln\frac{b}{t}\right]^{\alpha-1}\frac{|f(z)-f(t)|}{t}dt\right] \\
\leq & \frac{\alpha M}{\left[\ln\frac{b}{a}\right]^{\alpha}}\left[\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{|x-t|}{t}dt+\int_{G_1}^{G_2}\left[\ln\frac{G_2}{t}\right]^{\alpha-1}\frac{|y-t|}{t}dt\right. \\
& \left.+\int_{G_2}^{b}\left[\ln\frac{b}{t}\right]^{\alpha-1}\frac{|z-t|}{t}dt\right] \\
= & \frac{\alpha M G_{\alpha,\lambda,\eta,\mu}(x,y,z)}{\left[\ln\frac{b}{a}\right]^{\alpha}}.
\end{aligned}$$

Here, by making simple calculations, the integrals

$$\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{|x-t|}{t}dt,\quad \int_{G_1}^{G_2}\left[\ln\frac{G_2}{t}\right]^{\alpha-1}\frac{|y-t|}{t}dt,\quad \int_{G_2}^{b}\left[\ln\frac{b}{t}\right]^{\alpha-1}\frac{|z-t|}{t}dt$$

can be obtained as following:

i. If $G_1 \leq G_2 \leq x \leq y \leq z$ or $G_1 \leq x \leq G_2 \leq y \leq z$, then

$$\begin{aligned}
\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}\frac{|x-t|}{t}dt &= \frac{x}{\alpha}\lambda^{\alpha}\left[\ln\frac{b}{a}\right]^{\alpha}-\int_a^{G_1}\left[\ln\frac{t}{a}\right]^{\alpha-1}dt \\
\int_{G_1}^{G_2}\left[\ln\frac{G_2}{t}\right]^{\alpha-1}\frac{|y-t|}{t}dt &= \frac{y}{\alpha}\eta^{\alpha}\left[\ln\frac{b}{a}\right]^{\alpha}-\int_{G_1}^{G_2}\left[\ln\frac{G_2}{t}\right]^{\alpha-1}dt \\
\int_{G_2}^{b}\left[\ln\frac{b}{t}\right]^{\alpha-1}\frac{|z-t|}{t}dt &= \frac{z}{\alpha}\mu^{\alpha}\left[\ln\frac{b}{a}\right]^{\alpha}-\frac{2z}{\alpha}\left[\ln\frac{b}{z}\right]^{\alpha} \\
& \quad -\int_{G_2}^{z}\left[\ln\frac{b}{t}\right]^{\alpha-1}dt+\int_{z}^{b}\left[\ln\frac{b}{t}\right]^{\alpha-1}dt.
\end{aligned}$$

ii. If $G_1 \leq x \leq y \leq G_2 \leq z$, then

$$\begin{aligned}
\int_a^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} \frac{|x-t|}{t} dt &= \frac{x}{\alpha}\lambda^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \int_a^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} dt \\
\int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} \frac{|y-t|}{t} dt &= \frac{y}{\alpha}\eta^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \frac{2y}{\alpha}\left[\ln \frac{G_2}{y}\right]^\alpha \\
&\quad - \int_{G_1}^{y} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt + \int_y^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt \\
\int_{G_2}^{b} \left[\ln \frac{b}{t}\right]^{\alpha-1} \frac{|z-t|}{t} dt &= \frac{z}{\alpha}\mu^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \frac{2z}{\alpha}\left[\ln \frac{b}{z}\right]^\alpha \\
&\quad - \int_{G_2}^{z} \left[\ln \frac{b}{t}\right]^{\alpha-1} dt + \int_z^{b} \left[\ln \frac{b}{t}\right]^{\alpha-1} dt.
\end{aligned}$$

iii. If $G_1 \leq x \leq y \leq z \leq G_2$, then

$$\begin{aligned}
\int_a^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} \frac{|x-t|}{t} dt &= \frac{x}{\alpha}\lambda^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \int_a^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} dt \\
\int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} \frac{|y-t|}{t} dt &= \frac{y}{\alpha}\eta^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \frac{2y}{\alpha}\left[\ln \frac{G_2}{y}\right]^\alpha \\
&\quad - \int_{G_1}^{y} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt + \int_y^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt \\
\int_{G_2}^{b} \left[\ln \frac{b}{t}\right]^{\alpha-1} \frac{|z-t|}{t} dt &= -\frac{z}{\alpha}\mu^\alpha \left[\ln \frac{b}{a}\right]^\alpha + \int_{G_2}^{b} \left[\ln \frac{b}{t}\right]^{\alpha-1} dt.
\end{aligned}$$

iv. If $x \leq G_1 \leq G_2 \leq y \leq z$, then

$$\begin{aligned}
\int_a^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} \frac{|x-t|}{t} dt &= \frac{x}{\alpha}\lambda^\alpha \left[\ln \frac{b}{a}\right]^\alpha + \frac{2x}{\alpha}\left[\ln \frac{x}{\alpha}\right]^{\alpha-1} \\
&\quad - \int_a^{x} \left[\ln \frac{t}{a}\right]^{\alpha-1} dt + \int_x^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} dt \\
\int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} \frac{|y-t|}{t} dt &= \frac{y}{\alpha}\eta^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt \\
\int_{G_2}^{b} \left[\ln \frac{b}{t}\right]^{\alpha-1} \frac{|z-t|}{t} dt &= -\frac{z}{\alpha}\mu^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \int_{G_2}^{z} \left[\ln \frac{b}{t}\right]^{\alpha-1} dt \\
&\quad + \int_z^{b} \left[\ln \frac{b}{t}\right]^{\alpha-1} dt.
\end{aligned}$$

v. If $x \le G_1 \le y \le G_2 \le z$, then

$$\begin{aligned}
\int_a^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} \frac{|x-t|}{t} dt &= \frac{x}{\alpha}\lambda^\alpha \left[\ln \frac{b}{a}\right]^\alpha + \frac{2x}{\alpha}\left[\ln \frac{x}{\alpha}\right]^{\alpha-1} \\
&\quad - \int_a^x \left[\ln \frac{t}{a}\right]^{\alpha-1} dt + \int_x^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} dt \\
\int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} \frac{|y-t|}{t} dt &= \frac{y}{\alpha}\eta^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \frac{2y}{\alpha}\left[\ln \frac{G_2}{y}\right]^\alpha \\
&\quad - \int_{G_1}^y \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt + \int_y^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt \\
\int_{G_2}^b \left[\ln \frac{b}{t}\right]^{\alpha-1} \frac{|z-t|}{t} dt &= -\frac{z}{\alpha}\mu^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \int_{G_2}^z \left[\ln \frac{b}{t}\right]^{\alpha-1} dt \\
&\quad + \int_z^b \left[\ln \frac{b}{t}\right]^{\alpha-1} dt.
\end{aligned}$$

vi. If $x \le G_1 \le y \le z \le G_2$, then

$$\begin{aligned}
\int_a^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} \frac{|x-t|}{t} dt &= \frac{x}{\alpha}\lambda^\alpha \left[\ln \frac{b}{a}\right]^\alpha + \frac{2x}{\alpha}\left[\ln \frac{x}{\alpha}\right]^{\alpha-1} \\
&\quad - \int_a^x \left[\ln \frac{t}{a}\right]^{\alpha-1} dt + \int_x^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} dt \\
\int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} \frac{|y-t|}{t} dt &= \frac{y}{\alpha}\eta^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \frac{2y}{\alpha}\left[\ln \frac{G_2}{y}\right]^\alpha \\
&\quad - \int_{G_1}^y \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt + \int_y^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt \\
\int_{G_2}^b \left[\ln \frac{b}{t}\right]^{\alpha-1} \frac{|z-t|}{t} dt &= -\frac{z}{\alpha}\mu^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \int_{G_2}^b \left[\ln \frac{b}{t}\right]^{\alpha-1} dt.
\end{aligned}$$

vii. If $x \le y \le G_1 \le G_2 \le z$, then

$$\begin{aligned}
\int_a^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} \frac{|x-t|}{t} dt &= \frac{x}{\alpha}\lambda^\alpha \left[\ln \frac{b}{a}\right]^\alpha + \frac{2x}{\alpha}\left[\ln \frac{x}{\alpha}\right]^{\alpha-1} \\
&\quad - \int_a^x \left[\ln \frac{t}{a}\right]^{\alpha-1} dt + \int_x^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} dt \\
\int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} \frac{|y-t|}{t} dt &= -\frac{y}{\alpha}\eta^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt \\
\int_{G_2}^b \left[\ln \frac{b}{t}\right]^{\alpha-1} \frac{|z-t|}{t} dt &= -\frac{z}{\alpha}\mu^\alpha \left[\ln \frac{b}{a}\right]^\alpha - \int_{G_2}^z \left[\ln \frac{b}{t}\right]^{\alpha-1} dt \\
&\quad + \int_z^b \left[\ln \frac{b}{t}\right]^{\alpha-1} dt.
\end{aligned}$$

viii. If $x \leq y \leq G_1 \leq z \leq G_2$ or $x \leq y \leq z \leq G_1 \leq G_2$, then

$$\begin{aligned}
\int_a^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} \frac{|x-t|}{t} dt &= \frac{x}{\alpha}\lambda^\alpha \left[\ln \frac{b}{a}\right]^\alpha + \frac{2x}{\alpha}\left[\ln \frac{x}{\alpha}\right]^{\alpha-1} \\
&\quad - \int_a^x \left[\ln \frac{t}{a}\right]^{\alpha-1} dt + \int_x^{G_1} \left[\ln \frac{t}{a}\right]^{\alpha-1} dt \\
\int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} \frac{|y-t|}{t} dt &= -\frac{y}{\alpha}\eta^\alpha \left[\ln \frac{b}{a}\right]^\alpha + \int_{G_1}^{G_2} \left[\ln \frac{G_2}{t}\right]^{\alpha-1} dt \\
\int_{G_2}^{b} \left[\ln \frac{b}{t}\right]^{\alpha-1} \frac{|z-t|}{t} dt &= -\frac{z}{\alpha}\mu^\alpha \left[\ln \frac{b}{a}\right]^\alpha + \int_{G_2}^{b} \left[\ln \frac{b}{t}\right]^{\alpha-1} dt.
\end{aligned}$$

□

Under the assumptions of Theorem 2.1, we obtain the following results.

Corollary 2.2. *If we take $\alpha = 1$ in Theorem 2.1, the we get the following general integral inequality:*

$$\begin{aligned}
&\left|\lambda f(x) + \eta f(y) + \mu f(z) - \frac{1}{\ln \frac{b}{a}} \int_a^{G_1} \frac{f(t)}{t} dt + \int_{G_1}^{G_2} \frac{f(t)}{t} dt + \int_{G_2}^{b} \frac{f(t)}{t} dt\right| \\
= &\left|\lambda f(x) + \eta f(y) + \mu f(z) - \frac{1}{\ln \frac{b}{a}} \int_a^{b} \frac{f(t)}{t} dt\right| \\
\leq &\frac{M}{\ln \frac{b}{a}} \left[\int_a^{G_1} \frac{|x-t|}{t} dt + \int_{G_1}^{G_2} \frac{|y-t|}{t} dt + \int_{G_2}^{b} \frac{|z-t|}{t} dt\right]. \qquad (2.1)
\end{aligned}$$

The general integral inequality given Corollary 2.2 is reduced to the different types of integral inequalities given in the literature in special cases. These inequalities are given in the following results.

Corollary 2.3. *If we take $\lambda = \frac{1}{2}, \eta = \frac{1}{2}, \mu = 0$ and $x = a, y = b$ (in this case $G_1 = \sqrt{ab} = G$, $G_2 = b$) in the equality (2.1), then we obtain the following Trapezoid type inequality*

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(t)}{t} dt\right| &\leq \frac{M}{\ln \frac{b}{a}} \left[\int_a^G \frac{t-a}{t} dt + \int_G^b \frac{b-t}{t} dt\right] \\
&= \frac{M}{\ln \frac{b}{a}} \left[\frac{b-a}{2} \ln \frac{b}{a} + 2G - a - b\right] \qquad (2.2)
\end{aligned}$$

which reduces the inequality (19) in [8, Corollary 10]

Corollary 2.4. *If we take $\lambda = 1, \eta = \mu = 0$ (in this case $G_1 = b = G_2$) in the equality (2.1), then we obtain the following Ostrowski type inequality:*

$$\begin{aligned}
\left|f(x) - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(t)}{t} dt\right| &\leq \frac{M}{\ln \frac{b}{a}} \left[\int_a^x \frac{x-t}{t} dt + \int_x^b \frac{t-x}{t} dt\right] \\
&= \frac{M}{\ln \frac{b}{a}} \left[x \ln\left(\frac{x^2}{ab}\right) + (a+b-2x)\right]. \qquad (2.3)
\end{aligned}$$

In this inequality, specially if we choose $x = \sqrt{ab}$*, then we obtain the following midpoint type inequality:*

$$\left| f(\sqrt{ab}) - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(t)}{t} dt \right| \leq \frac{M}{\ln \frac{b}{a}} \left[(a + b - 2\sqrt{ab}) \right] \tag{2.4}$$

which reduces the inequality (16) in [8, Corollary 9]

Corollary 2.5. *If we take* $\lambda = \frac{1}{4} = \mu$*,* $\eta = \frac{1}{2}$*,* $x = a, z = b, y = \sqrt{ab} = G$ *in the equality (2.1), then we obtain the following Bullen type inequality:*

$$\begin{aligned} & \left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\sqrt{ab}\right) \right] - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(t)}{t} dt \right| \\ \leq & \frac{M}{\ln \frac{b}{a}} \left[\int_a^{G_1} \frac{t-a}{t} dt + \int_{G_1}^{G_2} \frac{|G-t|}{t} dt + \int_{G_2}^b \frac{b-t}{t} dt \right] \\ = & \frac{M}{\ln \frac{b}{a}} \left[2 \left(\sqrt[4]{a^3 b} + \sqrt[4]{ab^3} - \sqrt{ab} - \frac{a+b}{2} \right) + \frac{b-a}{4} \ln \frac{b}{a} \right] \end{aligned} \tag{2.5}$$

Corollary 2.6. *If we take* $\lambda = \frac{1}{6} = \mu$*,* $\eta = \frac{2}{3}$*,* $x = a, z = b, y = \sqrt{ab} = G$ *in the equality (2.1), then we obtain the following Simpson type inequality:*

$$\begin{aligned} & \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\sqrt{ab}\right) \right] - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(t)}{t} dt \right| \\ \leq & \frac{M}{\ln \frac{b}{a}} \left[2 \left(\sqrt[6]{a^5 b} + \sqrt[6]{ab^5} - \sqrt{ab} - \frac{a+b}{2} \right) + \frac{b-a}{6} \ln \frac{b}{a} \right]. \end{aligned} \tag{2.6}$$

3. Application to Special Means

Let us recall the following special means of two positive number a, b with $b > a$:

(1) The arithmetic mean

$$A = A(a, b) := \frac{a+b}{2}.$$

(2) The geometric mean

$$G = G(a, b) := \sqrt{ab}.$$

(3) The harmonic mean

$$H = H(a, b) := \frac{2ab}{a+b}.$$

(4) The logarithmic mean

$$L = L(a, b) := \frac{b-a}{\ln b - \ln a}.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq A.$$

To prove the results of this section, we need the following lemma:

Lemma 3.1 ([22])**.** *Let* $f : [a, b] \to \mathbb{R}$ *be differentiable with* $\|f'\|_\infty < \infty$*. Then* f *is an* M*-Lipschitzian function on* $[a, b]$ *where* $M = \|f'\|_\infty$ *.*

Proposition 3.2. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$|A(a^n, b^n) - L(a^n, b^n)| \leq nb^{n-1}\left[\frac{b-a}{2} + \frac{2}{\ln\frac{b}{a}}\left[G(a,b) - A(a,b)\right]\right].$$

Proof. The proof follows by the inequality (2.2) applied for the Lipschitzian function $f(x) = x^n$ on the interval $[a, b]$. □

Proposition 3.3. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$|x^n - L(a^n, b^n)| \leq \frac{2nb^{n-1}}{\ln\frac{b}{a}}\left[x\left(\ln x - \ln G(a,b) - 1\right) + A(a,b)\right].$$

Proof. The proof follows by the inequality (2.3) applied for the Lipschitzian function $f(x) = x^n$ on the interval $[a, b]$. □

Proposition 3.4. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$\begin{aligned}&\left|\frac{1}{2}\left[A(a^n, b^n) + G^n(a,b)\right] - L(a^n, b^n)\right| \\ \leq\ & nb^{n-1}\left[\frac{2}{\ln\frac{b}{a}}\left(\sqrt[4]{a^3b} + \sqrt[4]{ab^3} - G(a,b) - A(a,b)\right) + \frac{b-a}{4}\right].\end{aligned}$$

Proof. The proof follows by the inequality (2.5) applied for the Lipschitzian function $f(x) = x^n$ on the interval $[a, b]$. □

Proposition 3.5. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$\begin{aligned}&\left|\frac{1}{3}\left[A(a^n, b^n) + 2G^n(a,b)\right] - L(a^n, b^n)\right| \\ \leq\ & nb^{n-1}\left[\frac{2}{\ln\frac{b}{a}}\left(\sqrt[6]{a^5b} + \sqrt[6]{ab^5} - G(a,b) - A(a,b)\right) + \frac{b-a}{6}\right].\end{aligned}$$

Proof. The proof follows by the inequality (2.6) applied for the Lipschitzian function $f(x) = x^n$ on the interval $[a, b]$. □

Proposition 3.6. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$\left|H^{-1}(a,b) - L(a,b)\,G^{-2}(a,b)\right| \leq \frac{1}{a^2}\left[\frac{b-a}{2} + \frac{2}{\ln\frac{b}{a}}\left[G(a,b) - A(a,b)\right]\right].$$

Proof. The proof follows by the inequality (2.2) applied for the Lipschitzian function $f(x) = \frac{1}{x}$ on the interval $[a, b]$. □

Proposition 3.7. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$\left|\frac{1}{x} - L(a,b)\,G^{-2}(a,b)\right| \leq \frac{2}{a^2\ln\frac{b}{a}}\left[x\left(\ln x - \ln G(a,b) - 1\right) + A(a,b)\right].$$

Proof. The proof follows by the inequality (2.3) applied for the Lipschitzian function $f(x) = \frac{1}{x}$ on the interval $[a, b]$. □

Proposition 3.8. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$\begin{aligned} &\left|\frac{1}{2}\left[H^{-1}(a,b)+G^{-1}(a,b)\right]-L(a,b)G^{-2}(a,b)\right| \\ \leq\ & \frac{1}{a^2}\left[\frac{2}{\ln\frac{b}{a}}\left(\sqrt[4]{a^3b}+\sqrt[4]{ab^3}-G(a,b)-A(a,b)\right)+\frac{b-a}{4}\right]. \end{aligned}$$

Proof. The proof follows by the inequality (2.5) applied for the Lipschitzian function $f(x)=\frac{1}{x}$ on the interval $[a,b]$. □

Proposition 3.9. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$\begin{aligned} &\left|\frac{1}{3}\left[H^{-1}(a,b)+2G^{-1}(a,b)\right]-L(a,b)G^{-2}(a,b)\right| \\ \leq\ & \frac{1}{a^2}\left[\frac{2}{\ln\frac{b}{a}}\left(\sqrt[6]{a^5b}+\sqrt[6]{ab^5}-G(a,b)-A(a,b)\right)+\frac{b-a}{6}\right]. \end{aligned}$$

Proof. The proof follows by the inequality (2.6) applied for the Lipschitzian function $f(x)=\frac{1}{x}$ on the interval $[a,b]$. □

Proposition 3.10. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$0 \leq \frac{1}{a}\left[\frac{b-a}{2}+\frac{2}{\ln\frac{b}{a}}\left[G\left(a,b\right)-A\left(a,b\right)\right]\right].$$

Proof. The proof follows by the inequality (2.2) applied for the Lipschitzian function $f(x)=\ln x$ on the interval $[a,b]$. □

Proposition 3.11. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$\left|\ln x-\ln G(a,b)\right| \leq \frac{2}{a\ln\frac{b}{a}}\left[x\left(\ln x-\ln G\left(a,b\right)-1\right)+A(a,b)\right].$$

Proof. The proof follows by the inequality (2.3) applied for the Lipschitzian function $f(x)=\ln x$ on the interval $[a,b]$. □

Proposition 3.12. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$0 \leq \frac{1}{a}\left[\frac{2}{\ln\frac{b}{a}}\left(\sqrt[4]{a^3b}+\sqrt[4]{ab^3}-G(a,b)-A(a,b)\right)+\frac{b-a}{4}\right].$$

Proof. The proof follows by the inequality (2.5) applied for the Lipschitzian function $f(x)=\ln x$ on the interval $[a,b]$. □

Proposition 3.13. *For $b > a > 0$ and $n \geq 1$, we have the following inequality:*

$$0 \leq \frac{1}{a}\left[\frac{2}{\ln\frac{b}{a}}\left(\sqrt[6]{a^5b}+\sqrt[6]{ab^5}-G(a,b)-A(a,b)\right)+\frac{b-a}{6}\right].$$

Proof. The proof follows by the inequality (2.6) applied for the Lipschitzian function $f(x)=\ln x$ on the interval $[a,b]$. □

4. Applications to the quadrature formula

Assume $\wp$ is a division of the interval $[a,b]$ such that

$$\wp: \quad a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

and let define

$$\ln \wp: \quad \ln a = \ln x_0 < \ln x_1 < ... < \ln x_{n-1} < \ln x_n = \ln b$$

For a given function $f : [a,b] \to \mathbb{R}$ we consider the quadrature formula

$$\begin{aligned} \int_{\ln a}^{\ln b} (f \circ \exp)(x)dx &= \int_a^b \frac{f(x)}{x}dx \\ &= T(f \circ \exp, \ln \wp) + E(f \circ \exp, \ln \wp), \end{aligned}$$

where

$$T(f \circ \exp, \ln \wp) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (\ln x_{i+1} - \ln x_i)$$

for the trapezoidal version and

$$T(f \circ \exp, \ln \wp) = \sum_{i=0}^{n-1} f\left(\sqrt{x_i x_{i+1}}\right) (\ln x_{i+1} - \ln x_i)$$

for the midpoint version and $E(f \circ \exp, \ln \wp)$ denotes the associated approximation error of the integral $\int_a^b (f(x)/x)\, dx$.We derive some new error estimates as follows.

Proposition 4.1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and the function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a M-Lipschitzian function on I. then for each division $\wp$ of the interval $[a,b]$ trapezoidal error satisfies,*

$$|E(f \circ \exp, \ln \wp)| \leq M \sum_{i=0}^{n-1} \left[\frac{x_{i+1} - x_i}{2} \ln \frac{x_{i+1}}{x_i} + 2\sqrt{x_i x_{i+1}} - x_i - x_{i+1} \right]. \quad (4.1)$$

and midpoint error satisfies

$$|E(f \circ \exp, \ln \wp)| \leq M \sum_{i=0}^{n-1} \left[x_i + x_{i+1} - 2\sqrt{x_i x_{i+1}} \right].$$

Proof. We apply the inequality (2.2) on the sub-intervals $[x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$ given by the division $\wp$. Adding from $i = 0$ to $i = n-1$ we deduce

$$\left| \frac{f(x_i) + f(x_{i+1})}{2} \ln \frac{x_{i+1}}{x_i} - \int_{x_i}^{x_{i+1}} \frac{f(t)}{t} dt \right| \leq M \left[\frac{x_{i+1} - x_i}{2} \ln \frac{x_{i+1}}{x_i} + 2\sqrt{x_i x_{i+1}} - x_i - x_{i+1} \right]. \quad (4.2)$$

Summing over i from 0 to $n-1$, we deduce that

$$|E(f \circ \exp, \ln \wp)| \leq M \sum_{i=0}^{n-1} \left[\frac{x_{i+1} - x_i}{2} \ln \frac{x_{i+1}}{x_i} + 2\sqrt{x_i x_{i+1}} - x_i - x_{i+1} \right].$$

Similarly, by usin the inequality (2.4) we deduce that

$$|E(f \circ \exp, \ln \wp)| \leq M \sum_{i=0}^{n-1} \left[x_i + x_{i+1} - 2\sqrt{x_i x_{i+1}} \right]$$

for midpoint error. □

5. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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